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Compactness of Isoresonant Potentials

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Compactness of Isoresonant Potentials

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

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ABSTRACT OF DISSERTATION

Compactness of Isoresonant Potentials

Brüning considered sets of isospectral Schrödinger operators with smooth real potentials on a compact manifold of dimension three. He showed the set of potentials associated to an isospectral set is compact in the topology of smooth functions by relating the spectrum to the trace of the heat semi-group. Similarly, we can consider the resonances of Schrodinger operators with real valued potentials on Euclidean space of whose support lies inside a ball of fixed radius that generate the same resonances as some fixed Schrödinger operator, an "isoresonant" set of potentials. This isoresonant set of potentials is also compact in the topology of smooth functions for dimensions one and three. The basis of the result stems from the relation of a regularized wave trace to the resonances via the Poisson formula (also known as the Melrose trace formula). The second link is the small-t asymptotic expansion of the regularized wave trace whose coefficients are integrals of the potential function and its derivatives. For an isoresonant set these coefficients are equal due to the Poisson formula. The equivalence of coefficients allows us to uniformly bound the potential functions and their derivatives with respect to the isoresonant set. Finally, taking a sequence of functions in the isoresonant set we use the uniform bounds to construct a convergent subsequence using the Arzelà-Ascoli theorem.

KEYWORDS: compact, wave trace, heat trace, isospectral ,resonance, Schródinger operator

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Compactness of Isoresonant Potentials

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Dedicated to my wife, Gina Marie Wolf.

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Chapter 1 Introduction

Schrödinger operators are a topic of interest in both functional analysis and mathematical physics. We begin by considering the setting of a d-dimensional smooth Riemannian manifold M with a fixed metric g. If we pair the Laplace-Beltrami operator $-\Delta_g$ and a real-valued smooth potential function $V \in C_0^\infty(M)$ then can look at the Schrödinger operator $H_V = -\Delta_g + V$.

We can characterize the operator and thus the potential function with the spectrum of the operator, $\sigma(H_V)$, which consists of all $z \in \mathbb{C}$ such that $H_V - z$ is not invertible. If M is compact then $\sigma(H_V)$ consists of only real eigenvalues with finite multiplicities and an accumulation point at infinity. Fixing a reference potential $V_0 \in C^\infty(M)$ lets us define an isopectral set of potential functions, $\mathrm{Iso}(V_0)$, as the set of all potentials V such that $\sigma(H_V) = \sigma(H_{V_0})$ (including multiplicity).

This characterization allows for discussion about properties of these isopectral sets. Brüning [2] showed that for low dimension that such an isopectral set is compact with the following theorem.

Theorem 1.0.1. *(Brüning [2]) Let M be a smooth compact manifold with $d \leq 3$ and fixed metric g. Fix $V_0 \in C^\infty(M)$ and consider the operator*

$$H_{V_0} = -\Delta_g + V_0.$$

The set

$$\{V \in C^\infty(M) \,|\, \sigma(H_V) = \sigma(H_{V_0})\}$$

is compact in the C^∞ topology.

Theorem 1.0.1 shows that for any isospectral sequence there will be a convergent subsequence in a Fréchet metric generated by the Sobolev semi-norms. Brüning uses the trace of the heat semi-group as the main tool in his proof to relate the eigenvalues of H_V to a small t expansion involving the Sobolev semi-norms of V.

A basic example of such a family is the rotations of any fixed potential on the sphere, S^2. Another notable family of such isospectral functions are the solutions to the Korteweg-de Vries equation on S^1 [2]. A smooth potential on $V_0 \in C^\infty(S^1)$ can be used as the initial data and flowed along KdV to yield a one parameter family of smooth potential functions $\{V_t(x)\}$. The Lax pair property of the KdV then gives that $\sigma(H_{V_0}) = \sigma(H_{V_t})$ for all t.

A natural attempt to extend this result is to move the setting from a compact manifold M to $\mathbb{R}^d$. There are several roadblocks that occur when trying to make this transition. The first is that the spectrum, $\sigma(H_V)$, no longer consist only of eigenvalues but also contains essential spectrum from $[0, \infty)$. A second problem is the heat invariants play a central role in the compactness argument, and the heat semi-group is not longer trace class on $\mathbb{R}^d$.

The first problem will be addressed chapter 4 by finding an analog for the eigenvalues in resonances, which are the poles of the meromorphic continuation of $R_{H_V}(\lambda) =$

$(H_V - \lambda^2)^{-1}$ in the lower half plane. Note that here we have set $z = \lambda^2$ and thus $R_{H_V}(\lambda)$ is meromorphic in the upper half plane with the poles being the square roots of eigenvalues lying on the positive imaginary axis. This will necessitate using the trace of the wave group instead of the trace of the heat semi-group as there is a relation between the trace of the wave group and resonances through the Poisson formula.

This leads to addressing the issue of the trace class of the heat semi-group and wave group. In both cases it can be shown that the regularized heat or wave operator are both trace class on $\mathbb{R}^d$. In chapter 3 we will derive a representation of the regularized heat trace from using the work of Hitrik and Polterovich [13]. This will be useful as we can relate the heat invariants to the wave invariants in odd dimension through the use of the wave to heat transform.

In chapter 5 we'll have enough tools available to implement the Brüning/Donnelly mechanics and prove the central theorem.

Theorem 1.0.2. *Let $V \in C_0^\infty(\mathbb{R}^d)$ with $d = 1, 3$, the operator*

$$H_V = -\Delta + V,$$

and fix $r > 0$ and $V_0 \in C_0^\infty(B_r(0))$. Then the set

$$\{V \in C_0^\infty(B_r(0)) \,|\, H_V \text{ is isoresonant to } H_{V_0}.\}$$

is compact in the C^∞ topology.

Chapter 2 Background: An Overview of the Isospectral Case

We begin by reviewing compactness results for smooth real potentials of isospectral Schrödinger operators on a smooth compact manifold without boundary. We will first give a definition for an isospectral set of potentials followed by a short discussion of the heat semi-group and its trace. We will then give a statement and proof of the compactness result given by Theorem 1.0.1.

2.1 Isospectral Set

We define $H_V = -\Delta_g + V$ to be the Schrödinger Operator with potential living on the smooth compact d dimensional Riemannian manifold M with a fixed metric g. The operator $-\Delta_g$ is the positive Laplace-Beltrami operator whose representation in local coordinates is give by

$$-\Delta_g = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x_i}g^{ij}\sqrt{\det g}\frac{\partial}{\partial x_j}$$

while the potential V is an element of $C_0^\infty(M)$.

Since M is a compact d dimensional manifold, then $\sigma(H_V)$ consists only of real, countable eigenvalues of finite multiplicity with a single accumulation point at infinity. For example, Let $M = S^2$ then the eigenvalues of $-\Delta$ are given by the formula $\lambda = l(l+1)$ for all $l \in \mathbb{N}_0$ with multiplicity $2l+1$.

We can now define what it means for two potentials to be isospectral.

Definition 2.1.1. *Isospectral Set*

Let M be a smooth compact d dimensional manifold without boundary with fixed metric g and fix $V_0 \in C^\infty(M)$. We define the isospectral set of V_0 to be

$$\mathrm{Iso}(V_0) = \{V \in C^\infty(M) | \sigma(H_V) = \sigma(H_{V_0})\}.$$

2.2 Heat Semi-Group

If we consider the heat equation given by

$$\partial_t u + H_V u = 0 \tag{2.1}$$

with initial data $u(x,0) = u_0(x)$. Then we can construct a solution using the one parameter semi-group, $e^{-H_V t}$, that sends a solution to the heat equation at time t_0 to a solution at time $t_0 + t$. In particular,

$$u(x,t) = e^{-H_V t}u_0(x) \tag{2.2}$$

solves Equation 2.1. We call $e^{-H_V t}$ the heat semi-group or the heat operator.

2.3 Heat Trace

If we naively wanted to consider the trace of the operator H_V we would quickly see that for $\{\lambda_i\} = \sigma(H_V)$

$$\lim_{k\to\infty} \sum_{i=1}^{k} \lambda_i = \infty.$$

However if we consider the trace of the heat semi-group the functional calculus gives that the eigenvalues of $e^{-H_V t}$ are given by $e^{-\lambda_i t}$ where $\lambda_i \in \sigma(H_V)$

We then get that the trace is given by

$$Tr(e^{-H_V t}) = \lim_{k\to\infty} \sum_{i=1}^{k} e^{-\lambda_i t} \tag{2.3}$$

which converges for every $t > 0$. We then consider what happens as t goes to zero.

Gilkey [1] gives that

$$Tr(e^{-H_V t}) \sim (4\pi t)^{-\frac{d}{2}} \sum_{j=0}^{\infty} a_j t^j \qquad \text{as } t \to 0. \tag{2.4}$$

The a_j's depend on the potential V, the metric g, and their derivatives. The coefficents are given in Brüning [2] and Donnelly [3] as

$$\begin{aligned}
a_0 &= Vol(M) \\
a_1 &= \int_M \left(V + \frac{K}{3}\right) \\
a_2 &= \int_M \left(V^2 + Vf(D^\alpha g) + h(D^\alpha g)\right) \\
a_j &= c_j \int_M |\nabla^{j-2} V|^2 + \sum_{k=0}^{j} \sum_{\substack{\alpha\in\mathbb{N}_0^k \\ |\alpha|=l(k)}} \int_M P^k_{\alpha_1}(V) P^k_{\alpha_2}(V) \cdots P^k_{\alpha_k}(V)
\end{aligned} \tag{2.5}$$

where K is the curvature and f and h are functions of the derivatives of the metric g. The last formula being used for $j \geq 3$. The terms $P^k_{\alpha_i}$ are non-linear differential operators acting on V with coefficients dependent on the derivatives of the the metric. We can observe the non-linearity in the a_0, a_1, and a_3 coefficients when we set $V = 0$. The constraint $l(k) \leq 2(j-k)$. The $P^k_{\alpha_i}$'s also have the following constraints,

$$\begin{aligned}
Ord\, P^k_{\alpha_i} &\leq (j-k) \\
\sum_{i=1}^{k} Ord\, P^k_{\alpha_i} &\leq 2(j-k).
\end{aligned} \tag{2.6}$$

These bounds on the $Ord\, P^k_{\alpha_i}$ are an improvement by Donnelly [3] over Brüning's [2] bounds of $j-3$ and $2(j-3)$ respectively.

2.4 Uniform Boundedness of Isospectral Potential Semi-Norms.

In order to show compactness in the C^∞ topology we will need uniform bounds in each of the Sobolev semi-norms for all V in our isospectral set. We do this by assuming we have uniform bounds on the $j-3$ norm and then using the small t asymptotics to show uniform bounds on the $j-2$ norm.

Theorem 2.4.1. *Let M be a smooth d-dimensional compact manifold and fix $V_0 \in C^\infty(M)$. Suppose $\|V\|_{m,2} \leq C_m$ for some $m > \frac{d}{2} - 2$ and for all $V \in Iso(V_0)$. Then $\|V\|_{m,2} \leq C_m$ for all m and $V \in Iso(V_0)$. Furthermore, for $d \leq 3$, $\|V\|_{m,2} \leq C_m$ for all m.*

The last line of the theorem is evident from the fact that $0 > \frac{d}{2} - 2$ for $d \leq 3$, so we only need a bound on the L^2 norm of V. Applying Cauchy-Schwarz to the a_2 term of Equation 2.5 gives

$$\|V\|_2^2 = a_2 - \int_M (Vf(D^\alpha g) + h(D^\alpha g)) \leq C + \|V\|_2 \|f\|_2 \tag{2.7}$$

which implies $\|V\|_2$ is bounded as it has the form $x^2 - bx - c \leq 0$ with fixed $b, c \geq 0$. For the first piece of the theorem we will need to consider the remaining heat invariants given again by the formula

$$a_j = c_j \int |\nabla^{j-2} V|^2 + \sum_{k=0}^{j} \sum_{\substack{\alpha \in \mathbb{N}_0^k \\ |\alpha| \leq l(k)}} \int P^k_{\alpha_1}(V) P^k_{\alpha_2}(V) \cdots P^k_{\alpha_k}(V) \tag{2.8}$$

where $P^k_{\alpha_i}$ is a differential operator on V and the term $l(k) \leq 2(j-k)$. We also recall the constraints $Ord\, P^k_{\alpha_i} \leq \min\{j-3, j-k\}$ and $\sum_{i=1}^{k} Ord\, P^k_{\alpha_i} \leq \min\{2(j-3), 2(j-k)\}$.

If we assume an apriori bound on the $j-3$ norm then a rearrangement gives that for each j we have the bound F

$$\|V\|_{j-2,2}^2 \leq C \left(1 + \sum_{k=0}^{j} \sum_{\substack{\alpha \in \mathbb{N}_0^k \\ |\alpha| \leq l(k)}} \int |P^k_{\alpha_1}(V) P^k_{\alpha_2}(V) \cdots P^k_{\alpha_k}(V)| \right)$$

where $C > 0$ is independent of our choice of V from the isospectral set. The strategy is to then show each term

$$\int |P^k_{\alpha_1}(V) P^k_{\alpha_2}(V) \cdots P^k_{\alpha_k}(V)|$$

is bounded by a constant independent of our choice of V ($d = 1$) or by a multiple of $1 + \|V\|_{j-2,2}^{\beta}$ with $\beta < 2$ ($d \geq 3$). Together these bounds yield a uniform bound on $\|V\|_{j-2,2}$ for each j. Furthermore, the bounding of this integral over the product of $P_{\alpha_i}^k$'s is what necessitates the condition $m > \frac{d}{2} - 2$ in the theorem.

We begin with the case $d = 1$. Here we will use 2.4.1 in conjunction with Hölder's inequality to obtain our result.

Lemma 2.4.1. *Let $u \in C_0^1(\mathbb{R})$ then*

$$\|u\|_\infty \leq C\|u\|_{1,2}$$

Proof. This follows from Theorem 6.0.3 with $d = 1$.

□

Proposition 2.4.1. *Let $d = 1$ and $\|V\|_{j-3,2} \leq M$, $V \in \mathrm{IsoRes}(V_0)$ then*

$$\int |P_{\alpha_1}^k(V)P_{\alpha_2}^k(V)\cdots P_{\alpha_k}^k(V)| \leq C$$

Where C depends only on M and j

Proof: We use the bounds on the order of the $P_{\alpha_i}^k$ terms to conclude that there are at most 2 terms with order greater than $j-4$, and those terms have order $j-3$. Lemma 2.4.1 will then allow us to get the desired bounds.

Case 1: Assume 0 terms of order $j-3$

Then using the Lemma for each i

$$|P_{\alpha_i}^k(V)| \leq C\|P_{\alpha_i}^k(V)\|_{1,2} \leq C\|V\|_{j-3,2}$$

This gives

$$\int |P_{\alpha_1}^k(V)P_{\alpha_2}^k(V)\cdots P_{\alpha_k}^k(V)| \leq C^k M^k \leq C^j M^j$$

(WLOG we may assume $C, M \geq max(1, m(B_r))$)

Case 2: Assume 1 term is of order $j-3$

Using the results from Case 1 with the Hölder inequality we have

$$\begin{aligned}\int |P_{\alpha_1}^k(V)P_{\alpha_2}^k(V)\cdots P_{\alpha_k}^k(V)| &\leq C^{k-1}M^{k-1}\int |P_{\alpha_1}^k(V)| \\ &\leq C^{k-1}M^{k-1}m(B_r)\|V\|_{j-3,2} \\ &\leq C^j M^j\end{aligned} \tag{2.9}$$

Case 3: Assume 2 terms of order $j-3$

Again using the results of Case 1 and the Hölder inequality gives

$$
\begin{aligned}
\int |P^k_{\alpha_1}(V)P^k_{\alpha_2}(V)\cdots P^k_{\alpha_k}(V)| &\leq C^{k-2}M^{k-2}\int |P^k_{\alpha_1}(V)P^k_{\alpha_2}(V)| \\
&\leq C^{k-2}M^{k-2}\|V\|^2_{j-3,2} \\
&\leq C^jM^j
\end{aligned} \tag{2.10}
$$

□

Next we consider the case $d = 3$ and use a proof given by Donnelly [3] which requires reordering the $P^k_{\alpha_i}(V)$ terms. Fix k and use the truncated notation $P_i = P^k_{\alpha_i}(V)$. Then reorder the terms based on the $ord\, P_i$ and define T in the following way

$$T = P_1P_2\cdots P_lP_{l+1}\cdots P_k$$

where the ordering and l are chosen s.t.

$$
\begin{aligned}
i \leq l &\Rightarrow d > 2(j-3-ord\, P_i) \\
i > l &\Rightarrow d \leq 2(j-3-ord\, P_i)
\end{aligned} \tag{2.11}
$$

We will separate $\int|T|$ using the generalized Hölder's inequality and apply the Sobolev embedding theorem to get an estimate. Note that the conditions imposed by this reordering determines which case of the Sobolev embedding theorem for $p = 2$ and $k = (j - ord\, P_i - 3)$ is appropriate.

Proposition 2.4.2. *(Lemma 4.6, Donnelly [3]) If $d \geq 3, j > \frac{d}{2}+1$, and $\|V\|_{j-3,2} \leq C_1$, then*

$$\int |T| \leq C_2\|V\|^{\beta}_{j-2,2}$$

where $\beta < 2$ and C_2 depends on C_1

Proof. We will look at the possible values of l and for each case the general strategy will be to use the generalized Hölder's inequality to show

$$\int |T| \leq C\prod_{i=1}^{k}\|P_i\|_{r_i}$$

with $\sum\limits_{i=1}^{k}\frac{1}{r_i} = 1$.

For $i > l$ we use two Sobolev inequalities. When $d < 2(j-3-ord\, P_i)$ we will use embedding for L^∞ [Theorem 6.0.3],

$$\|P_i\|_\infty \leq C_1(1 + \max_{|\alpha \leq ord\, P_i}\|D^\alpha V\|_\infty) \leq C_2\|V\|_{j-3,2}.$$

and when when $d = 2(j-3-ord\, P_i)$ we will use

$$\|P_i\|_{r_i} \leq C_1 \|V\|_{ord\,P_i, r_i} \leq C_2 \|V\|_{j-3,2}$$

where $2 \leq r_i < \infty$ [Theorem 6.0.4]. These two inequalities give the bound

$$\int |T| \leq C \prod_{i=1}^{k} \|P_i\|_{r_i} \leq \tilde{C} \|V\|_{j-3,2}^{k-l} \prod_{i=1}^{l} \|P_i\|_{r_i}$$

The remainder of the proof is to show that for $1 \leq i \leq l$ we can choose r_i to apply the appropriate Sobolev inequality.

Case 0: When $l = 0$ the estimate holds for $\beta = 0$ using the above method.

Case 1: Letting $l = 1$ implies $d > 2(j - ord\,P_1 - 3)$, so setting

$$r_1 = \frac{2d}{d - 2(j - ord\,P_1 - 3)}$$

yields

$$\|P_1\|_{r_1} \leq C \|P_1\|_{j - ord\,P_1 - 3, 2} \leq C \|V\|_{j-3,2}$$

by the generalized Sobolev inequality. The only condition on j is $2 \leq \frac{2d}{d-2(j-ord\,P_1-3)}$ or $j - 3 \geq ord\,P_1$ which is true for every P_i and j.

Since $\frac{1}{r_1} \leq \frac{1}{2}$, we can choose the remaining r_i's to meet the condition $\sum_{i=1}^{k} \frac{1}{r_i} = 1$. So for $l = 1$ we have the bound with $\beta = 0$.

Case 2: Assume $l = 2$.

If $k = l = 2$ then we know for $i = 1, 2$ that $ord\,P_i \leq j - 3$, so Hölder's inequality gives

$$\int |P_1 P_2| \leq \|P_1\|_2 \|P_2\|_2 \leq C \|V\|_{j-3,2}^2$$

Assuming $k > 2$, if $ord\,P_1$ and $ord\,P_2$ are such that r_1 and r_2 (as chosen in case $l = 1$) satisfy

$$\frac{1}{r_1} + \frac{1}{r_2} < 1$$

then we proceed as in the case $l = 1$ and apply the generalized Hölder's inequality to get the result with $\beta = 0$.

Now assume $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Since $ord\,P_i \leq j-3$, this implies $ord\,P_1 = ord\,P_2 = j-3$ and thus $r_1 = r_2 = 2$. We may then apply the generalized Hölder's inequality to get.

$$\int |T| \leq C \|P_1\|_{r_1+\varepsilon} \prod_{i=2}^{k} \|P_i\|_{r_i}$$

Where $\varepsilon > 0$ and r_i for $i > 3$ are chosen to satisfy the Hölder condition. Furthermore if we choose ε such that $r_1 + \varepsilon < \frac{2d}{d-2}$ then the general Sobolev inequality gives that

$$\|P_1\|_{r_1+\varepsilon_i} \leq C_1\|P_1\|_{1,2} \leq C_2\|V\|_{j-2,2}$$

So we get the result with $\beta = 1$.

Case 3: Assume $l \geq 3$ and $d > 2(j - ord\, P_i - 2)$ for $i = 1, 2$

Let r_i be as in case 1 and 2 and set

$$s_i = \frac{2d}{d - 2(j - ord\, P_i - 2)}$$

L_p interpolation gives that for $0 < \varepsilon_i < 1$ there exists a $0 < \beta_i < 1$ s.t.

$$\|P_i\|_{r_i+\varepsilon_i} \leq \|P_i\|_{r_i}^{\beta_i}\|P_i\|_{s_i}^{1-\beta_i}$$

Thus using the generalized Hölder's inequality we have

$$\begin{aligned}
\int |T| &\leq C\|P_1\|_{r_1+\varepsilon_1}\|P_2\|_{r_2+\varepsilon_2}\prod_{i=3}^{k}\|P_i\|_{r_i} \\
&\leq \|P_1\|_{r_1}^{\beta_1}\|P_1\|_{s_1}^{1-\beta_1}\|P_2\|_{r_2}^{\beta_2}\|P_2\|_{s_2}^{1-\beta_2}\prod_{i=3}^{k}\|P_i\|_{j-ord\, P_i-3,2} \\
&\leq C\|V\|_{j-3,2}^{\beta_1}\|P_1\|_{s_1}^{1-\beta_1}\|V\|_{j-3}^{\beta_1}\|P_2\|_{s_2}^{1-\beta_2}\prod_{i=3}^{k}\|V\|_{j-3,2} \\
&\leq C\|P_1\|_{s_1}^{1-\beta_1}\|P_2\|_{s_2}^{1-\beta_2} \\
&\leq C\|V\|_{j-2,2}^{\beta}
\end{aligned} \tag{2.12}$$

Where $\beta < 2$. r_i may be chosen arbitrarily for $i > l$ and as in case 1 for $i \leq l$, so in order to satisfy the Hölder condition we require.

$$\frac{1}{r_1+\varepsilon_1} + \frac{1}{r_2+\varepsilon_2} + \sum_{i=3}^{l}\frac{1}{r_i} < 1$$

Which, for sufficiently large $\varepsilon_1, \varepsilon_2 < 1$ is implied by

$$\frac{1}{s_1} + \frac{1}{s_2} + \sum_{i=3}^{l}\frac{1}{r_i} < 1$$

Substituting for s_i and r_i gives

$$\sum_{i=1}^{2} \frac{d-2(j-ord\, P_i-2)}{2d} + \sum_{i=3}^{l} \frac{d-2(j-ord\, P_i-3)}{2d} < 1$$

which may be rewritten as

$$(d-2j-6)l + 2\sum_{i=1}^{l} ord\, P_i < 2d+4$$

Because $\sum_{i=1}^{l} ord\, P_i \leq 2(j-k)$ it is sufficient to show

$$(d-2j-6)l + 4(j-k) < 2d+4$$

Using assumption $l \geq 3$ lets us rewrite the inequality as

$$\frac{d}{2} + 3 - \frac{2k-4}{l-2} < j$$

Then $k \geq l \geq 3$ gives $\frac{2k-4}{l-2} \geq \frac{2k-4}{k-2} = 2$ so it suffices for

$$\frac{d}{2} + 1 < j$$

Case 4: $l \geq 3$ and $d \leq 2(j - ord\, P_i - 2)$ for P_1 and P_2.

For $2 \leq s < \infty$ we have the embedding

$$\|P_i\|_s \leq C\|P_i\|_{j-ord\, P_i-2,2} \leq C\|V\|_{j-2,2}.$$

Then L_p interpolation gives for $2 < t < s$

$$\|P_i\|_t \leq \|P_i\|_s^{\beta_i} \|P_i\|_2^{1-\beta_i}.$$

We may take t to be arbitrarily large reducing the Hölder condition to

$$\sum_{i=3}^{l} \frac{1}{r_i} < 1.$$

If $l = 3$ the condition is met as $r_3 \geq 2$, so we assume $l \geq 4$. Substituting for r_i and rewriting the inequality we get

$$(l-2)(d-2j+6) + 2\sum_{i=3}^{l} ord\, P_i < 2d.$$

Using the inequality $\sum_{i=3}^{l} ord\, P_i \leq \sum_{i=1}^{k} ord\, P_i \leq 2(j-k)$ gives the sufficient condition

$$(l-2)(d-2j+6) + 4(j-k) < 2d$$

which can be recast as

$$(l-4)d+6(l-2)-4k<(2l-8)j.$$

If $l=4$ then the inequality reduces to $12-4k<0$ which always holds as $4=l\leq k$. For $l\geq 5$ we rewrite the inequality as

$$\frac{d}{2}+3\frac{l-4}{l-4}+3\frac{2}{l-4}-\frac{2k}{l-4}<j$$

which reduces to

$$\frac{d}{2}+3-\frac{(2k-6)}{l-4}<j.$$

Since $k\geq l$ it is sufficient for the following series of inequalities to hold:

$$\frac{d}{2}+3-\frac{(2k-6)}{k-4}<j$$

$$\frac{d}{2}+1-\frac{(2)}{k-4}<j \tag{2.13}$$

$$\frac{d}{2}+1<j.$$

This gives the condition on j.

□

2.5 Compactness of Isospectral Sets

In order to prove the compactness result we still need to show that an arbitrary sequence $\{V_i\}\subset \mathrm{Iso}(V_0)$ has a convergent subsequence.

Definition 2.5.1. *Let $\mathcal{V}_{r,d}$ be the set*

$$\mathcal{V}_{r,d}=\{V\in C_0^\infty(\mathbb{R}^d)|\ supp\ V\subset B_r(0),\|V\|_{W_{j,2}}<C_j\,,\forall j\}$$

Theorem 2.4.1 gives that $\mathrm{Iso}(V_0)\subset\mathcal{V}_{r,d}$ for suitable choice of r and d. It will be shown in section 5.2 that $\mathcal{V}_{r,d}$ is compact. It then only remains to be shown that the limit potential is still in the isospectral set.

Proposition 2.5.1. *Let M a smooth compact manifold, and fix $V_0\in C_0^\infty(M)$. Let $\{V_i\}\subset Iso(V_0)$ s.t. $V_i\to V$ in $C_0^\infty(M)$. Then $V\in Iso(V_0)$.*

Proof:

Let A be a self-adjoint operator then $\lambda\in\sigma(A)$ iff there exists a sequence $u_i\subset D(A)$ s.t. $\|u_i\|=1$ and $\|(A-\lambda)u_i\|\to 0$ [9]. So we only need come up with such a sequence.

Let $\lambda_0 \in \sigma(H_{V_0})$ then for each i there is a $u_i \in ker(H_{V_i} - \lambda_0)$ s.t. $u_i \neq 0$ with $\|u_i\|_2 = 1$

However,

$$(-\Delta + V - \lambda_0)u_i = (-\Delta + V - V_i + V_i - \lambda_0)u_i = [V - V_i]u_i$$

which implies

$$\|(-\Delta + V - \lambda_0)u_i\|_2 \leq \|V - V_i\|_\infty \|u\|_2$$

so $\|(-\Delta + V - \lambda_0)u_i\|_2 \to 0$ as $i \to \infty$. Thus $\lambda_0 \in \sigma(H_v)$

Now suppose $\lambda_0 \in \sigma(H_V)$. If for $\varepsilon > 0$ there is a $u \in D(A)$ such that $\|(A-\lambda)u\| \leq \varepsilon\|u\|$ then A has spectrum inside the interval $[\lambda-\varepsilon, \lambda+\varepsilon]$ [9]. Choose $u \in ker(H_V - \lambda_0)$ and let $\varepsilon > 0$ be given. Then $\exists N$ s.t. $i > N$ implies $\|V - V_i\|_\infty \leq \varepsilon$. Thus,

$$\begin{aligned} \|(H_{V_i} - \lambda_0)u\|_2 &= \|(H_V - \lambda_0 + V - V_i)u\|_2 \\ &= \|(V - V_i)u\|_2 \\ &\leq \|V - V_i\|_\infty \|u\|_2 \\ &\leq \varepsilon\|u\|_2 \end{aligned} \tag{2.14}$$

So $\sigma(H_{V_0}) \cap [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] \neq \emptyset$. This is true for each $\varepsilon > 0$ which implies $\lambda_0 \in \sigma(H_{V_0})$. Therefore we get that $\sigma(H_V) = \sigma(H_{V_0})$, which gives the final result of Theorem 1.0.1 restated below.

Theorem 2.5.1. *(Brüning [2]/Donnelly [3]) Let M be a smooth compact manifold with $d \leq 3$ and fixed metric g. Fix $V_0 \in C^\infty(M)$ and consider the operator*

$$H_{V_0} = -\Delta_g + V_0.$$

The set

$$\{V \in C^\infty(M) \,|\, \sigma(H_V) = \sigma(H_{V_0})\}$$

is compact in the C^∞ topology.

Chapter 3 Heat Trace Expansions in $\mathbb{R}^d$

In this chapter we will show that the regularized heat operator is trace class, define a regularized heat trace, and give an explicit form to the terms in the heat invariants. This new form for the heat invariants will be useful in determining the form of the wave invariants presented in chapter 4 as the they are related through the wave to heat transform.

3.1 Heat Trace Expansions in $\mathbb{R}^d$

The Gilkey [1] [3] formula give the heat invariants on a compact manifold as

$$a_j = c_j \int |\nabla^{j-2} V|^2 + \sum_{k=0}^{j} \sum_{\substack{\alpha \in \mathbb{N}_0^k \\ |\alpha| \le l(k)}} \int P^k_{\alpha_1}(V) P^k_{\alpha_2}(V) \cdots P^k_{\alpha_k}(V)$$

Where the terms $P^k_{\alpha_i}$ are differential operators acting on the potential V and derivatives of the metric, g, of the manifold. Similarly, the constraint $l(k)$ is affected by the derivatives of the metric. If we consider the regularized heat trace in $\mathbb{R}^d$ we explicitly determine the terms $P^k_{\alpha_i}$ and $l(k)$.

We then ask the question if similar asymptotics can be derived for $\mathbb{R}^d$. Unfortunately the heat semi-group $e^{-H_V t}$ is not trace class on $\mathbb{R}^d$. Instead we must regularize the heat operator in order to take a trace and obtain small t asymptotics.

Proposition 3.1.1. *The operator $e^{-tH_V} - e^{-tH_0}$ is trace class on $\mathbb{R}^d$ for $d \le 3$.*

Proof. Using Duhamel's formula we write the operator $e^{-tH_V} - e^{-tH_0}$ as

$$\begin{aligned}
e^{-tH_V} - e^{-tH_0} &= \left(e^{-tH_V} e^{tH_0} - I\right) e^{-tH_0} \\
&= \int_0^t \frac{d}{ds}\left[e^{-sH_V} e^{sH_0}\right] ds e^{-tH_0} \\
&= \int_0^t -H_v e^{-sH_V} e^{sH_0} + e^{-sH_V} H_0 e^{sH_0} ds e^{-tH_0} \\
&= \int_0^t e^{-sH_V} [H_0 - H_V] e^{sH_0} ds e^{-tH_0} \\
&= -\int_0^t e^{-sH_V} V e^{-(t-s)H_0} ds
\end{aligned} \tag{3.1}$$

We now use Duhamel's formula to analyze the operator in the trace norm, $\|\cdot\|_{Tr}$. Note that the trace norm is more typically written as $\|\cdot\|_1$, however this notation will be reserved for the L^p norms. The same will be true for the Hilbert-Schmidt norm.

$$\begin{aligned}\|e^{-tH_V} - e^{-tH_0}\|_{Tr} &= \|\int_0^t e^{-sH_V} V e^{-(t-s)H_0} ds\|_{Tr} \\ &\leq \int_0^t \|e^{-sH_V}\chi_{supp\,V} V e^{-(t-s)H_0}\|_1 \, ds\end{aligned} \tag{3.2}$$

The integrand then has the following estimates where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm:

$$\|e^{-sH_V}\chi_{supp\,V} V e^{-(t-s)H_0}\|_{Tr} \leq \|e^{-sH_V}\chi_{supp\,V}\|_{HS}\|V e^{-(t-s)H_0}\|_{HS} \tag{3.3}$$

The integral kernel for $Ve^{-(t-s)H_0}$ is $\frac{1}{(4\pi(t-s))^{\frac{d}{2}}} e^{\frac{-|x-y|^2}{4(t-s)}} V(y)$.So,

$$V e^{-(t-s)H_0} f(x) = \frac{1}{(4\pi(t-s))^{\frac{d}{2}}} \int_{\mathbb{R}^d} V(y) e^{\frac{-|x-y|^2}{4(t-s)}} f(y)\, dy \tag{3.4}$$

Simon [?] gives that

$$\begin{aligned}\|V e^{-(t-s)H_0}\|_{HS}^2 &= \|\frac{1}{(4\pi(t-s))^{\frac{d}{2}}} e^{\frac{-|x-y|^2}{4(t-s)}} V(y)\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^d)} \\ &= \int \frac{1}{(4\pi(t-s))^d} e^{\frac{-|x-y|^2}{2(t-s)}} V^2(y)\, dxdy \\ &= \frac{\|V\|_2^2}{(4\pi(t-s))^d} \int e^{\frac{-|x|^2}{2(t-s)}}\, dx \\ &= \frac{2^{\frac{d}{2}}\|V\|_2^2}{(4\pi)^d (t-s)^{\frac{d}{2}}} \int e^{-|z|^2}\, dz \\ &= \frac{\|V\|_2^2}{2^{\frac{3d}{2}}(\pi(t-s))^{\frac{d}{2}}}\end{aligned} \tag{3.5}$$

Using the change of coordinates $z = \frac{x}{(2(t-s))^{\frac{1}{2}}}$. Thus we get:

$$\|V e^{-(t-s)H_0}\|_{HS} = \frac{\|V\|_2}{2^{\frac{3d}{4}}(\pi(t-s))^{\frac{d}{4}}}. \tag{3.6}$$

Let $K_{V,s}(x,y)$ be the integral kernel of e^{-sH_V} then applying the result by Simon gives:

$$\|e^{-sH_V}\chi_{supp\,V}\|_{HS}^2 = \|K_{V,s}(x,y)\chi_{supp\,V}\|_2^2 \tag{3.7}$$

We then use the upper bound on elliptic operators for $L^2(\mathbb{R}^d)$ given by Davies[4] as

$$K_{V,s}(x,y) \leq \frac{c}{s^{\frac{d}{2}}} e^{\frac{-b|x-y|^2}{s}} \tag{3.8}$$

Where c and $b < 1$ are positive constants. This gives for $E = m(supp\, V)$:

$$\begin{aligned} \|e^{-sH_V}\chi_{supp\,V}\|_{HS}^2 &\leq \frac{c^2}{s^d}\int e^{\frac{-2b|x-y|^2}{s}}\chi_{supp\,V}(y)\,dxdy \\ &= \frac{E^2c^2}{s^d}\int e^{\frac{-2b|x|^2}{s}}\,dx \\ &= \frac{E^2c^2}{(2bs)^{\frac{d}{2}}}\int e^{-|z|^2}\,dz \\ &= \frac{E^2c^2\pi^{\frac{d}{2}}}{(2bs)^{\frac{d}{2}}}. \end{aligned} \tag{3.9}$$

This yields

$$\|e^{-sH_V}\chi_{supp\,V}\|_{HS}^2 \leq \frac{Ec\pi^{\frac{d}{4}}}{(2bs)^{\frac{d}{4}}}. \tag{3.10}$$

We then combine our two estimates to get

$$\begin{aligned} \|e^{-tH_V} - e^{-tH_0}\|_{Tr} &\leq \int_0^t \frac{\|V\|_2}{2^{\frac{3d}{4}}(\pi(t-s))^{\frac{d}{4}}}\frac{Ec\pi^{\frac{d}{4}}}{(2bs)^{\frac{d}{4}}}\,ds \\ &= \frac{Ec\|V\|_2}{2^d b^{\frac{d}{4}}}\int_0^t \frac{1}{(t-s)^{\frac{d}{4}}(s)^{\frac{d}{4}}}\,ds \\ &= \frac{Ec\|V\|_2}{2^d b^{\frac{d}{4}}}\frac{t}{t^{\frac{d}{2}}}\int_0^1 \frac{1}{(1-\lambda)^{\frac{d}{4}}(\lambda)^{\frac{d}{4}}}\,d\lambda \end{aligned} \tag{3.11}$$

Since $d < 3$ the integral in λ is finite and we have the bound:

$$\|e^{-tH_V} - e^{-tH_0}\|_{Tr} < CE\|V\|_2\frac{t}{t^{\frac{d}{2}}} \tag{3.12}$$

□

Definition 3.1.1. *Regularized Heat Trace*

Let $H_V = -\Delta + V$ with $H_0 = -\Delta$. Given the heat semi-group e^{-tH_V} we define the Regularized Heat Trace as

$$Tr(e^{-tH_V} - e^{-tH_0})$$

Lemma 3.1.1. *(Hitrik-Poltorovich) for $V \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$ the regularized heat trace on $\mathbb{R}^d$ is given by the formula*

$$Tr(e^{-tH_V} - e^{-tH_0}) \sim \frac{1}{(4\pi t)^{\frac{d}{2}}} \sum_{j=1}^{\infty} a_j t^j, \quad t \to 0^+$$

where

$$a_j = \int\limits_{\mathbb{R}^d} a_j(x) dx$$

$$a_j(x) = (-1)^j \sum_{m=0}^{j-1} \binom{j-1+\frac{d}{2}}{m+\frac{d}{2}} \frac{(-\Delta_y + V(y))^{m+j}(|x-y|^{2m})|_{y=x}}{4^m m!(m+j)!}$$

In the next proposition we will be using k-tuples of multi-indices with specific constraints. For this purpose we define the following set

$$\mathcal{A}_{j,k} = \left\{ \alpha = (\alpha^1, \cdots, \alpha^k) \;\middle|\; \begin{array}{c} \alpha^i \in \mathbb{N}_0^d \text{ for } 1 \le i \le k \\ |\alpha^i| \le j-k, \\ \sum_{i=1}^k |\alpha^i| = 2(j-k) \\ \sum_{i=1}^k \alpha_l^i \text{ is even for each } l. \end{array} \right\}$$

Proposition 3.1.2. *Let $V \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$ then the j^{th} coefficient of the regularized heat trace expansion, a_j, is given by*

$$\begin{aligned} a_1 &= -\int V \\ a_2 &= c_2 \int V^2 \\ a_3 &= c_3 \int |\nabla V|^2 + \tilde{c}_3 \int V^3. \end{aligned} \tag{3.13}$$

For $j > 3$ we have the general formula:

$$a_j = c_j \int |\nabla^{j-2} V|^2 + \sum_{k=3}^{j} \sum_{\alpha \in \mathcal{A}_{j,k}} c_\alpha \int D^{\alpha^1}(V) D^{\alpha^2}(V) \cdots D^{\alpha^k}(V). \tag{3.14}$$

Proof. We will use the Hitrik-Polterovich expansion for to get a formula for a_j's. Looking at the formula for $a_j(x)$ from 3.1.1 the first term a_1 is an immediate consequence of setting $j = 0$. For the terms a_2 and a_3 it will suffice to show the a_j general formula. We begin by considering the following pieced of the Hitrik-Polterovich formula:

$$(-\Delta_y + V(y))^{m+j}(|x-y|^{2m})|_{y=x}.$$

Fix $0 \leq k \leq m+j$ and define the function F_i^k as

$$F_i^k = \begin{cases} V(y) & i \leq k \\ -\Delta_y & i > k \end{cases}$$

then

$$(-\Delta_y + V(y))^{m+j} = \sum_{k=0}^{j+m} \sum_{\sigma \in S_{j+m}} \frac{1}{k!(j+m-k)!} F^k_{\sigma(1)} F^k_{\sigma(2)} \cdots F^k_{\sigma(j+m)}$$

where S_{j+m} is the symmetric group and the combinatorial term $k!(j+m-k)!$ is used to account that the potentials (and Laplacians) maybe reordered among themselves. We then want pass the Laplacians to the right using the Leibniz rule.

$$-\Delta V = (-\Delta V) - 2(\nabla V) \cdot \nabla - V\Delta$$

We will then be left with a sum of terms with a model term being

$$D^{\alpha^1} V D^{\alpha^2} V \cdots D^{\alpha^l} V D^{\beta}$$

where

$$|\alpha^1| + |\alpha^2| + \cdots + |\alpha^l| + |\beta| = 2(j+m-k)$$

We then look for a formula for $(D^\beta |x-y|^{2m})|_{y=x}$. In particular we want to know for which β is $(D^\beta |x-y|^{2m})|_{y=x} \neq 0$. First we observe that

$$\frac{\partial}{\partial y_i} |x-y|^{2m} = 2m|x-y|^{2m-2}(y_i - x_i)$$

so we conclude that if β_i is odd then $(D^\beta |x-y|^{2m})|_{y=x} = 0$ due to the remaining $(y_i - x_i)$ term. Next we observe that

$$\frac{\partial}{\partial y_j}\frac{\partial}{\partial y_i} |x-y|^{2m} = 2m(2m-2)|x-y|^{2m-4}(y_i - x_i)(y_j - x_i) + 2m|x-y|^{2m-2}\delta_{ij}.$$

Setting $i = j$ gives

$$\begin{aligned} \frac{\partial^2}{\partial y_i^2} |x-y|^{2m} &= 2m(2m-2)|x-y|^{2m-2} + 2m|x-y|^{2m-2} \\ &= (2m)(2m-1)|x-y|^{2m-2}. \end{aligned} \tag{3.15}$$

Induction on m then provides the following formula

$$(D_y^\beta |x-y|^{2m})|_{y=x} \begin{cases} (2m)! & |\beta| = 2m, \beta_i \text{ even } \forall i \\ 0 & \text{otherwise} \end{cases}$$

So the expansion is a sum of terms of the form

$$C D^{\alpha^1} V D^{\alpha^2} V \cdots D^{\alpha^k} V$$

where $\sum_{i=1}^{k} |\alpha^i| = 2(j-k)$ and $\sum_i \alpha^i_j$ is even for each j. Turning back to the a_j terms we see that they are a sum of integrals of the form

$$\int D^{\alpha^1} V D^{\alpha^2} V \cdots D^{\alpha^k} V$$

where integration by parts provides that we may assume $|\alpha^i| \le (j-k)$. We then may look at the various cases of k

Case $k=0$: this implies$|\beta| = j + m > 2m$ so the associated term is zero.

Case $k=1$: The integral

$$\int D^{\alpha} V = 0$$

as V is smooth and compactly supported.

Case $k=2$: Here we may assume then integration by parts and the fact that $\sum_i \alpha^i_j$ is even gives us $\alpha^1 = \alpha^2$

$$\int D^{\alpha^1} V D^{\alpha^2} V = \int |D^{\alpha^1} V|^2$$

Case $k \ge 3$ Again using integration by parts we have

$$\int D^{\alpha^1} V D^{\alpha^2} V \cdots D^{\alpha^k} V$$

with the added constraint that $|\alpha^i| \le j-k$ for each i. Thus the j^{th} coefficient in the regularized heat trace expansion has the form

$$a_j = c_j \int |\nabla^{j-2} V|^2 + \sum_{k=3}^{j} \sum_{\alpha \in \mathcal{A}_{j,k}} c_\alpha \int D^{\alpha^1}(V) D^{\alpha^2}(V) \cdots D^{\alpha^k}(V). \tag{3.16}$$

$\square$

Chapter 4 Resonances and the Wave Trace

This chapter will provide most of the tools and objects from which we will construct our compactness result in chapter 5. We will first discuss the resonances of H_V which are the analogs to the eigenvalues from the isospectral case. Next we will discuss the wave operator and a regularized wave trace and extract the wave invariants which are essentially the heat invariants. The well known Poisson formula will be used to relate the wave trace to the resonances.

4.1 Resonances

Fix a real valued $V \in C_0^\infty(\mathbb{R}^d)$, then for $\lambda \in \mathbb{C}$ with $Im\,\lambda > 0$ and $\lambda^2 \notin \sigma(H_V)$ we define the resolvent operator

$$R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1} \;: L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d). \tag{4.1}$$

The function R_V is operator valued and meromorphic on the upper half plane of $\mathbb{C}$. We can then extend R_V to all of $\mathbb{C}$ with the meromorphic continuation $\widetilde{R}_V$ where $\widetilde{R}_V|_{Im\,\lambda>0}(\lambda) = R_V(\lambda)$ and for $Im\,\lambda \leq 0$ $\widetilde{R}_V$ takes $L^2_{\text{comp}} \to H^2_{\text{loc}}$. The poles of $\widetilde{R}_V$ are called the *resonances* with multiplicity $m_{R_V}(\lambda)$ given by:

$$m_{\lambda_i}(V) = \text{Rank}\left[\frac{-1}{2\pi i}\int\limits_{\gamma} R_{H_V}(\lambda) 2\lambda d\lambda\right]. \tag{4.2}$$

where γ is a closed curve about λ_i containing no other resonances.

Definition 4.1.1. *Let $V \in C_0^\infty(\mathbb{R}^d)$ and define $res(V) = \{(\lambda_i, m_{R_V}(\lambda_i))\}$ where the ordered pair $(\lambda_i, m_{R_V}(\lambda_i))$ is a resonance associated with the Schrödinger operator $-\Delta + V$ and its multiplicity, $m_{R_V}(\lambda_i)$.*

4.2 Isoresonant Criteria

We now define a set of isoresonant potentials using 4.1.1.

Definition 4.2.1.

$$\text{IsoRes}(V_0, r, d) = \{V \in C_0^\infty(\mathbb{R}^d, \mathbb{R}) | supp\, V \subset B_r(0), res(V) = res(V_0)\}$$

Fixing $r > 0$ is necessary so that we may avoid translations of potentials (which will give not compact isoresonant sets) and maintain control over the size of the support of the potentials.

4.3 Compactness of the Isoresonant Set

The central theorem will be an analog of Theorem 1.0.1, which we will state here.

Theorem 4.3.1. *Let $r > 0$ be fixed, $d = 1, 3$ and $B_r(0) \subset \mathbb{R}^d$. Fix $V_0 \in C_0^\infty(B_r(0), \mathbb{R})$. Then the set* $\mathrm{IsoRes}(V_0, r, d)$ *is compact with respect to the C^∞ topology.*

In order to prove the theorem we will need to show that a sequence $\{V_n\} \subset \mathrm{IsoRes}(V_0, r, d)$ has a convergent subsequence in the C^∞ topology. We will do this by building a Frechet metric from the L^2 semi-norms of the derivatives, showing that $\mathrm{IsoRes}(V_0, r, d)$ is equicontinuous in every derivative, and then applying Arzaela-Ascoli to extract a convergent subsequence in the Frechet metric.

To show equicontinuity we will need uniform bounds on the L^∞ norms of each derivative, which we will acquire from uniform bounds on the L^2 norms of each derivative using an embedding theorem. These uniform L^2 bounds are a result of the invariance of the wave traces with respect to $\mathrm{IsoRes}(V_0, r, d)$ through the Poisson formula for resonances and the relation of the wave invariants to the heat invariants of the the operators $-\Delta + V_n$. The heat invariants are used to provide the bounds.

4.4 Strongly Continuous Wave Group

We begin the discussion of the wave trace with a short review of the wave group. Consider the equation

$$\partial_{tt} u + (-\Delta + V)u = 0 \tag{*}$$

with initial data $u(0) = u_0$ and $\partial_t u(0) = u_1$

Then $W_V(t)$ is the strongly continuous one parameter unitary group that takes the solution at time 0 to the solution at time t.

$$W_V(t)(u_0(x), u_1(x))^T = (u(x,t), \partial_t u(x,t))^T$$

The infinitesimal generator of $W_V(t)$ given by L_V.

$$\frac{d}{dt} W_V(t)(u_0(x), u_1(x))^T = \frac{d}{dt}(u(t,x), \partial_t u(t,x))^T$$

$$L_V(u(t,x), \partial_t u(t,x)) = (\partial_t u(t,x)), [\Delta - V(x)]\, u(t,x)) \tag{4.3}$$

$$L_V = \begin{bmatrix} 0 & 1 \\ \Delta - V & 0 \end{bmatrix}$$

We denote the case $V = 0$ as L_0. Then the operator $L_V - L_0$ is given by

$$L_V - L_0 = \begin{bmatrix} 0 & 0 \\ -V & 0 \end{bmatrix}. \tag{4.4}$$

The wave group, $W_V(t)$, is not a trace class operator, however once regularized $W_V(t) - W_0(t)$ is trace class in a sense of distribution. For any $\rho(t) \in C_0^\infty(\mathbb{R})$ and a sufficiently large s (independent of $\rho(t)$)

$$\int \rho(t) Tr\left(W_V(t) - W_0(t)\right)\, dt\, < C\|\rho\|_{s,2}.$$

The regularized wave trace also has a small-t asymptotic expansion. The coefficents in this expansion, labelled w_j, are the wave invariants. The following well known theorem is due to Melrose [7].

Theorem 4.4.1. *Let d be odd and $V \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$. Then the operator $W_V(t) - W_0(t)$ is trace class and has the small-t expansions:*

for $d = 1$

$$Tr(W_v(t) - W_0(t)) \sim \sum_{j=1}^{\infty} w_j |t|^{2j-d} \tag{4.5}$$

for $d \geq 3$ odd

$$Tr(W_v(t) - W_0(t)) \sim \sum_{j=1}^{\frac{d-1}{2}} w_j \delta^{(d-1-2j)}(t) + \sum_{j=\frac{d+1}{2}}^{\infty} w_j |t|^{2j-d}. \tag{4.6}$$

In Proposition 4.5.3 the wave invariants, w_j, will be shown to be a constant multiple of the heat invariants using the wave to heat transform.

4.4.1 Duhamel's principle

The integral kernel of $W_V(t)$ is not known, however we may use Duhamel's principle to expand the difference $W_V(t) - W_0(t)$ as a sum of integrals over $W_0(t)$, $L_V - L_0$ and a remainder term.

$$
\begin{aligned}
W_V(t) - W_0(t) &= [W_V(t)W_0(-t) - 1]\, W_0(t) \\
&= \int ds \frac{d}{ds} [W_V(s) W_0(t-s)] \\
&= \int ds \frac{d}{ds} [W_V(s)]\, W_0(t-s) + W_V(s) \frac{d}{ds} [W_0(t-s)] \\
&= \int ds\, L_V W_V(s) W_0(t-s) - W_V(s) L_0 W_0(t-s) \\
&= \int ds\, W_V(s) L_V W_0(t-s) - W_V(s) L_0 W_0(t-s) \\
&= \int ds\, W_V(s)\, [L_V - L_0]\, W_0(t-s)
\end{aligned} \tag{4.7}
$$

Recursion of the formula m times yields

$$
\begin{aligned}
&W_V(t) - W_0(t) = \\
&\quad \sum_{i=1}^{m} \int ds_i \cdots ds_1 W_0(s_1)\, [L_V - L_0] \prod_{j=1}^{i-1} \left(W_0(s_{j+1} - s_j)\, [L_V - L_0] \right) W_0(t - s_i) \\
&\quad + \int ds_{i+1} \cdots ds_1 W_V(s_1)\, [L_V - L_0] \prod_{j=1}^{i} \left(W_0(s_{j+1} - s_j)\, [L_V - L_0] \right) W_0(t - s_{i+1})
\end{aligned} \tag{4.8}
$$

4.4.2 Integral Kernel for the Wave Equation with Potential

Given initial data $u_0(x)$ and $u_1(x)$ there exists an integral kernel $K_V(x, y, t)$ satisfying ,

$$
\begin{aligned}
K_V(x, y, 0) &= 0 \\
\partial_t K_V(x, y, t)|_{t=0} &= \delta(x - y) \\
[\partial_t^2 + (-\Delta_x + V_x)](K_V(x, y, t) &= 0
\end{aligned} \tag{4.9}
$$

such that.

$$
u(x, t) = \int \ \partial_t K_V(x, y, t) u_0(y) dy + \int K_V(x, y, t) u_1(y) dy
$$

solves $(*)$.

Taking the Fourier transform of $(*)$ gives

$$\partial_{tt}\widehat{u}(\varepsilon) + |\varepsilon|^2\widehat{u}(\varepsilon) = -\widehat{Vu}(\varepsilon) \tag{$\widehat{*}$}$$

which gives the integral kernels $\widehat{K}_V(x,y,t), \partial_t\widehat{K}_V(x,y,t)$. For $V=0$ we have

$$\widehat{K}_0(\varepsilon,\eta,t) = \delta(\varepsilon+\eta)\frac{\sin|\varepsilon|t}{|\varepsilon|} \tag{4.10}$$

$$\partial_t\widehat{K}_0(\varepsilon,\eta,t) = \delta(\varepsilon+\eta)\cos|\varepsilon|t$$

This allows us to recover $K_0(x,y,t)$ and $\partial_t K_0(x,y,t)$ by taking the inverse transform.

4.4.3 Integral Kernel of the Wave Group

Given a the wave group $W_V(t)$ we consider the integral kernel $\mathcal{K}(x,y,t)$ such that

$$W_V(t)(u_0(x),u_1(x))^T = \int_{\mathbb{R}^d}\mathcal{K}_V(x,y,t)(u_0(y),u_1(y))^T dy = (u(x,t),\partial_t u(x,t))^T$$

We denote the fundamental solution as $K_V(x,y,t), \partial_t K_V(x,y,t)$ such that

$$u(x,t) = \int_{\mathbb{R}^d}\partial_t K_V(x,y,t)u_0(y) + K_V(x,y,t)u_1(y)dy$$

Thus

$$\partial_t u(x,t) = \int_{\mathbb{R}^d}\partial_{tt}K_V(x,y,t)u_0(y) + \partial_t K_V(x,y,t)u_1(y)dy$$

which gives

$$\mathcal{K}_V(x,y,t) = \begin{bmatrix} \partial_t K_V(x,y,t) & K_V(x,y,t) \\ (\Delta - V)K_V(x,y,t) & \partial_t K_V(x,y,t) \end{bmatrix}$$

4.5 Wave Trace

4.5.1 Wave Trace Expansion

In order to write down a formula for the regularized wave trace we will need a new operation $\star$ which is similar to, but not a convolution.

Definition 4.5.1. *We define the operation $\star$ as*

$$B_1 \star B_2(x,y) = \int B_1(x,z)B_2(z,y)dz$$

Proposition 4.5.1. *The operation $\star$ is associative.*

$$\begin{aligned}((B_1 \star B_2) \star B_3)(x,y) &= \int (B_1 \star B_2)(x,z_1) B_3(z_1,y)\, dz_1 \\ &= \int \int B_1(x,z_2) B_2(z_2,z_1) B_3(z_1,y)\, dz_2 dz_1 \\ &= \int B_1(x,z_2) \int B_2(z_2,z_1) B_3(z_1,y)\, dz_1 dz_2 \\ &= \int B_1(x,z_2)(B_2 \star B_3)(z_2,y) dz_2 \\ &= (B_1 \star (B_2 \star B_3))(x,y)\end{aligned} \tag{4.11}$$

Given the associativity of $\star$ we will use the notation $\prod\limits_{i=1}^{\star m} B_i$ to mean

$$\prod_{i=1}^{\star m} B_i = B_1 \star B_2 \star \cdots \star B_{m-1} \star B_m$$

We now consider the wave trace

$$\begin{aligned}Tr(W_V(t) - W_0(t)) &= Tr\left(\int_0^t ds\, W_V(s) \left[L_V - L_0\right] W_0(t-s) \right) \\ &= \int_0^t ds\, Tr\left(W_V(s) \left[L_V - L_0\right] W_0(t-s) \right)\end{aligned} \tag{4.12}$$

The m^{th} term from the recursion of Duhamel's forumula (Equation 4.8) is

$$\begin{aligned}&\int_0^t \int_0^{s_{m-1}} ds_1 \cdots ds_m \; Tr\left(W_0(s_1) \left[L_v - L_0\right] \prod_{i=1}^{m-1} \left[W_0(s_{i+1} - s_i) \left[L_v - L_0\right]\right] W_0(t - s_m) \right) \\ &\qquad z \\ &= \int_0^t \int_0^{s_{m-1}} ds_1 \cdots ds_m \int_{\mathbb{R}^d} dx\, tr\left(\mathcal{K}_m(x,x,s_i) \right)\end{aligned} \tag{4.13}$$

where $\mathcal{K}_m$ is the relevant integral kernel. We know the formula for $\mathcal{K}_0$ from subsection 4.4.3. Now define $K_0^i = K_0(s_{i+1} - s_i)$ and assume

$$\mathcal{K}_m = (-1)^m \begin{bmatrix} K_0^0 \star \left(\prod_{i=1}^{\star(m-1)} (VK_0^i)\right) \star (V\partial_t K_0^m) & K_0^0 \star \left(\prod_{i=1}^{\star(m-1)} (VK_0^i)\right) \star (VK_0^m) \\ \partial_t K_0^0 \star \left(\prod_{i=1}^{\star(m-1)} (VK_0^i)\right) \star (V\partial_t K_0^m) & \partial_t K_0^0 \star \left(\prod_{i=1}^{\star(m-1)} (VK_0^i)\right) \star (VK_0^m) \end{bmatrix}$$

Then

$$\begin{aligned}
\mathcal{K}_{m+1} &= \begin{bmatrix} \partial_t K_0^{-1} & K_0^{-1} \\ \Delta K_0^{-1} & \partial K_0^{-1} \end{bmatrix} \star \begin{bmatrix} 0 & 0 \\ -V & 0 \end{bmatrix} \mathcal{K}_m \\
&= (-1)^m \begin{bmatrix} \partial_t K_0^{-1} & K_0^{-1} \\ \Delta K_0^{-1} & \partial K_0^{-1} \end{bmatrix} \star \begin{bmatrix} 0 & 0 \\ -V & 0 \end{bmatrix} \\
&\quad \begin{bmatrix} K_0^0 \star \left(\prod_{i=1}^{\star(m-1)} (VK_0^i)\right) \star (V\partial_t K_0^m) & K_0^0 \star \left(\prod_{i=1}^{\star(m-1)} (VK_0^i)\right) \star (VK_0^m) \\ \partial_t K_0^0 \star \left(\prod_{i=1}^{\star(m-1)} (VK_0^i)\right) \star (V\partial_t K_0^m) & \partial_t K_0^0 \star \left(\prod_{i=1}^{\star(m-1)} (VK_0^i)\right) \star (VK_0^m) \end{bmatrix} \\
&= (-1)^{m+1} \begin{bmatrix} \partial_t K_0^{-1} & K_0^{-1} \\ \Delta K_0^{-1} & \partial K_0^{-1} \end{bmatrix} \\
&\quad \star \begin{bmatrix} 0 & 0 \\ VK_0^0 \star \prod_{i=1}^{\star(m-1)} (VK_0^i) \star (V\partial_t K_0^m) & VK_0^0 \star \prod_{i=1}^{\star(m-1)} (VK_0^i) \star (VK_0^m) \end{bmatrix} \\
&= (-1)^{m+1} \begin{bmatrix} K_0^{-1} \star \left(\prod_{i=1}^{\star(m-1)} (VK_0^i)\right) \star (V\partial_t K_0^m) & K_0^{-1} \star \left(\prod_{i=1}^{\star(m-1)} (VK_0^i)\right) \star (VK_0^m) \\ \partial_t K_0^{-1} \star \left(\prod_{i=1}^{\star(m-1)} (VK_0^i)\right) \star (V\partial_t K_0^m) & \partial_t K_0^{-1} \star \left(\prod_{i=1}^{\star(m-1)} (VK_0^i)\right) \star (VK_0^m) \end{bmatrix}
\end{aligned} \tag{4.14}$$

Renumbering the superscript i gives

$$\mathcal{K}_{m+1} = (-1)^{m+1} \begin{bmatrix} K_0^0 \star \left(\prod_{i=1}^{\star m} (VK_0^i)\right) \star (V\partial_t K_0^m) & K_0^0 \star \left(\prod_{i=1}^{\star m} (VK_0^i)\right) \star (VK_0^m) \\ \partial_t K_0^0 \star \left(\prod_{i=1}^{\star m} (VK_0^i)\right) \star (V\partial_t K_0^m) & \partial_t K_0^0 \star \left(\prod_{i=1}^{\star m} (VK_0^i)\right) \star (VK_0^m) \end{bmatrix}$$

Then the integrand of the m^{th} term is

$$tr(\mathcal{K}_m) = (-1)^m \left(K_0^0 \star \prod_{i=1}^{\star(m-1)} (VK_0^i) \star (V\partial_t K_0^m) + \partial_t K_0^0 \star \prod_{i=1}^{\star(m-1)} (VK_0^i) \star (VK_0^m) \right)$$

Yielding

$$
\begin{aligned}
&Tr(W_V(t) - W_0(t)) = \\
&\qquad \sum_{m=1}^{\infty}(-1)^m \int\limits_{\mathbb{R}^d} \int\limits_0^t \int\limits_0^{s_{m-1}} ds_1 \cdots ds_m \quad \Bigg(K_0^0 \star \prod_{i=1}^{\star(m-1)} (VK_0^i) \star (V\partial_t K_0^m) \\
&\qquad\qquad + \partial_t K_0^0 \star \prod_{i=1}^{\star(m-1)} (VK_0^i) \star (VK_0^m) \Bigg) (x,x) dx
\end{aligned}
\tag{4.15}
$$

4.5.2 Fourier Transform Formulas

The next section will require some results of how the Fourier transform interacts with the $\star$ operation. We will consider the Fourier transform with $V \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$

$$
(B_0 \star VB_1 \star \cdots \star VB_m)\widehat{\ }
$$

Lemma 4.5.1.

$$
\widehat{B_1 \star B_2}(\varepsilon, \eta) = \int_{\mathbb{R}^d} \widehat{B_1}(\varepsilon, -\phi)\widehat{B_2}(\phi, \eta) d\phi
$$

Proof.

$$
\begin{aligned}
\widehat{B_1 \star B_2}(\varepsilon, \eta) &= \int_{\mathbb{R}^{2n}} e^{-i\varepsilon \cdot x} e^{-i\eta \cdot y} B_1 \star B_2(x,y)\, dxdy \\
&= \int_{\mathbb{R}^{3n}} e^{-i\varepsilon \cdot x} e^{-i\eta \cdot y} B_1(x,z) B_2(z,y) dxdydz \\
&= \int_{\mathbb{R}^{7n}} e^{-i\varepsilon \cdot x} e^{-i\eta \cdot y} e^{i\mu \cdot x} e^{i\gamma \cdot y} e^{i\phi_1 \cdot z} e^{i\phi_2 \cdot z} \widehat{B_1}(\mu, \phi_1) \\
&\qquad\qquad \widehat{B_2}(\phi_2, \gamma) dxdydzd\gamma d\mu d\phi_1 d\phi_2 \\
&= \int_{\mathbb{R}^{7n}} e^{-i(\varepsilon - \mu)\cdot x} e^{-i(\eta - \gamma)\cdot y} e^{i(\phi_1 - \phi_2)\cdot z} \widehat{B_1}(\mu, \phi_1) \\
&\qquad\qquad \widehat{B_2}(\phi_2, \gamma) dxdydzd\gamma d\mu d\phi_1 d\phi_2 \\
&= \int_{\mathbb{R}^{4n}} \delta(\varepsilon - \mu)\delta(\eta - \gamma)\delta(\phi_1 - \phi_2) \widehat{B_1}(\mu, \phi_1) \\
&\qquad\qquad \widehat{B_2}(\phi_2, \gamma) d\gamma d\mu d\phi_1 d\phi_2 \\
&= \int_{\mathbb{R}^d} \widehat{B_1}(\varepsilon, -\phi_2)\widehat{B_2}(\phi_2, \eta) d\phi_2
\end{aligned}
\tag{4.16}
$$

$\square$

Lemma 4.5.2.

$$(B_0 \star VB_1 \star \cdots \star VB_m)\widehat{\ } = \int_{\mathbb{R}^{n-1}} \widehat{B_0}(\varepsilon, -\gamma_1)\widehat{B_1}(\gamma_1 - \theta_1, -\gamma_2)\cdots \widehat{B_m}(\gamma_m - \theta_m, \eta)\prod_{i=1}^{m} V(\theta_i)d\gamma_i d\theta_i$$

Proof. Proceeding by induction we assume

$$(B_0 \star VB_1 \star \cdots \star VB_{m-1})\widehat{\ } = \int_{\mathbb{R}^{(m-1)n}} \widehat{B_0}(\varepsilon, -\gamma_1)\widehat{VB_1}(\gamma_1, -\gamma_2)\cdots \widehat{VB_{m-1}}(\gamma_{m-1}, \eta)\prod_{i=1}^{m-1} d\gamma_i$$

Then recursion of the formula given by Proposition 1 yields

$$\begin{aligned}(B_0 \star VB_1 \star \cdots \star VB_m)\widehat{\ } &= \int_{\mathbb{R}^{(m-1)n}} \widehat{B_0}(\varepsilon, -\gamma_1)\widehat{VB_1}(\gamma_1, -\gamma_2)\cdots \\ &\qquad (VB_{-1} \star VB_m)\widehat{\ }(\gamma_{m-1}, \eta)\prod_{i=1}^{m-1} d\gamma_i \\ &= \int_{\mathbb{R}^{nm}} \widehat{B_0}(\varepsilon, -\gamma_1)\widehat{VB_1}(\gamma_1, -\gamma_2)\cdots \\ &\qquad \widehat{VB_{m-1}}(\gamma_{m-1}, -\gamma_m)\widehat{VB_m}(\gamma_m, \eta)\prod_{i=1}^{m} d\gamma_i\end{aligned} \tag{4.17}$$

Using the Fourier transform identity

$$\widehat{fg}(\varepsilon) = \int_{\mathbb{R}^d} f(\varepsilon - \theta)g(\theta)d\theta$$

we get

$$(B_0 \star VB_1 \star \cdots \star VB_m)\widehat{\ } = \int_{\mathbb{R}^{2n}} \widehat{B_0}(\varepsilon, -\gamma_1)\widehat{B_1}(\gamma_1 - \theta_1, -\gamma_2)\cdots \widehat{B_m}(\gamma_m - \theta_n, \eta)\prod_{i=1}^{m} V(\theta_i)d\gamma_i d\theta_i$$

$\square$

Proposition 4.5.2. *The m^{th} term in the recursion formula for $Tr(W_V(t) - W_0(t))$ is given by*

$$\prod_{k=1}^{m}\int_0^{s_{k+1}} ds_k \int_{\mathbb{R}^d} d\varepsilon \frac{\sin(|\varepsilon|(t - s_m + s_1))}{|\varepsilon|} \left(\int_{\mathbb{R}^{d(m-1)}} \hat{V}(-\sum_{j=1}^{m-1}\theta_j) \prod_{i=1}^{m-1} \frac{\sin(|\varepsilon - \sum_{j=1}^{i}\theta_j|(s_{i+1} - s_i))}{|\varepsilon - \sum_{j=1}^{i}\theta_j|}\hat{V}(\theta_i)d\theta_i\right)$$

with $s_{m+1} = t$.

Proof. Along with the formula from Proposition 2 we use the substitutions

$$\begin{aligned} \widehat{B}_0(\alpha,\beta) &= \partial_t \widehat{K}_0(\alpha,\beta,s_1) = \delta(\alpha+\beta)\cos(|\alpha|s_1) \\ \widehat{B}_i(\alpha,\beta) &= \widehat{K}_0(\alpha,\beta,s_i - s_{i+1}) = \frac{1}{|\alpha|}\delta(\alpha+\beta)\sin(|\alpha|(s_{i+1} - s_i)) \\ \widehat{B}_m(\alpha,\beta) &= \widehat{K}_0(\alpha,\beta,(t - s_1)) = \frac{1}{|\alpha|}\delta(\alpha+\beta)\sin(|\alpha|(t - s_m)) \end{aligned} \tag{4.18}$$

which gives

$$\int_{\mathbb{R}^{2dm}} \delta(\varepsilon - \gamma_1)\cos(|\varepsilon|s_1) \prod_{i=1}^{m} \frac{1}{|\gamma_i - \theta_i|}\delta(\gamma_i - \theta_i - \gamma_{i+1})\sin(|\gamma_i - \theta_i|(s_{i+1} - s_i))\hat{V}(\theta_i)d\gamma_i d\theta_i$$

with $\gamma_{d+1} = \eta$ and $s_{m+1} = t$. Next we integrate over the $\gamma_i' s$ and θ_n to get the string of equalities.

$$\gamma_1 = \varepsilon$$

$$\gamma_2 = \gamma_1 - \theta_1 = \varepsilon - \theta_1$$

$$\vdots$$

$$\gamma_i = \gamma_{i-1} - \theta_i = \varepsilon - \sum_{j=1}^{i} \theta_j \tag{4.19}$$

$$\vdots$$

$$\eta = \varepsilon - \sum_{j=1}^{m} \theta_j$$

Setting $\varepsilon = \eta$ and integrating over ε

$$\int_{\mathbb{R}^{md}} \cos(|\varepsilon|s_1) \frac{\sin(|\varepsilon|(t-s_m))}{|\varepsilon|} \hat{V}(-\sum_{j=1}^{m-1} \theta_j) \prod_{i=1}^{m-1} \frac{\sin(|\varepsilon - \sum_{j=1}^{i} \theta_j|(s_{i+1}-s_i))}{|\varepsilon - \sum_{j=1}^{i} \theta_j|} \hat{V}(\theta_i) d\theta_i d\varepsilon$$

We then integrate over the s_i's.

$$\prod_{k=1}^{m} \int_0^{s_{k+1}} ds_k \int_{\mathbb{R}^d} \cos(|\varepsilon|s_1) \frac{\sin(|\varepsilon|(t-s_n))}{|\varepsilon|} \hat{V}(-\sum_{j=1}^{m-1} \theta_j)$$
$$\prod_{i=1}^{m-1} \frac{\sin(|\varepsilon - \sum_{j=1}^{i} \theta_j|(s_{i+1}-s_i))}{|\varepsilon - \sum_{j=1}^{i} \theta_j|} \hat{V}(\theta_i) d\theta_i d\varepsilon$$

Similarly if we take

$$\widehat{B}_0(\alpha,\beta) = \widehat{K}_0(\alpha,\beta,(t-s_1)) = \frac{1}{|\alpha|}\delta(\alpha+\beta)\sin(|\alpha|s_1)$$

$$\widehat{B}_i(\alpha,\beta) = \widehat{K}_0(\alpha,\beta,s_i-s_{i+1}) = \frac{1}{|\alpha|}\delta(\alpha+\beta)\sin(|\alpha|(s_{i+1}-s_i)) \tag{4.20}$$

$$\widehat{B}_n(\alpha,\beta) = \partial_t\widehat{K}_0(\alpha,\beta,t-s_n) = \delta(\alpha+\beta)\cos(|\alpha|s_n)$$

We get

$$\prod_{k=1}^{m}\int_0^{s_{k+1}} ds_k \int_{\mathbb{R}^{md}} \cos(|\varepsilon|(t-s_m))\frac{\sin(|\varepsilon|s_1)}{|\varepsilon|}\hat{V}(-\sum_{j=1}^{m-1}\theta_j)$$
$$\prod_{i=1}^{m-1}\frac{\sin(|\varepsilon-\sum_{j=1}^{i}\theta_j|(s_{i+1}-s_i))}{|\varepsilon-\sum_{j=1}^{i}\theta_j|}\hat{V}(\theta_i)d\theta_i d\varepsilon$$

The full formula for the m^{th} term is then

$$\prod_{k=1}^{m}\int_0^{s_{k+1}} ds_k \int_{\mathbb{R}^d}$$
$$\left(\int_{\mathbb{R}^{d(m-1)}} \cos(|\varepsilon|s_1)\frac{\sin(|\varepsilon|(t-s_m))}{|\varepsilon|}\hat{V}(-\sum_{j=1}^{m-1}\theta_j)\prod_{i=1}^{m-1}\frac{\sin(|\varepsilon-\sum_{j=1}^{i}\theta_j|(s_{i+1}-s_i))}{|\varepsilon-\sum_{j=1}^{i}\theta_j|}\hat{V}(\theta_i)d\theta_i\right.$$
$$\left.+\int_{\mathbb{R}^{d(m-1)}} \cos(|\varepsilon|(t-s_m))\frac{\sin(|\varepsilon|s_1)}{|\varepsilon|}\hat{V}(-\sum_{j=1}^{m-1}\theta_j)\prod_{i=1}^{m-1}\frac{\sin(|\varepsilon-\sum_{j=1}^{i}\theta_j|(s_{i+1}-s_i))}{|\varepsilon-\sum_{j=1}^{i}\theta_j|}\hat{V}(\theta_i)d\theta_i\right)d\varepsilon \tag{4.21}$$

Factoring we get

$$\prod_{k=1}^{m}\int_0^{s_{k+1}} ds_k \int_{\mathbb{R}^d} d\varepsilon \left(\cos(|\varepsilon|s_1)\frac{\sin(|\varepsilon|(t-s_m))}{|\varepsilon|}+\cos(|\varepsilon|(t-s_m))\frac{\sin(|\varepsilon|s_1)}{|\varepsilon|}\right)$$
$$\left(\int_{\mathbb{R}^{d(m-1)}}\hat{V}(-\sum_{j=1}^{m-1}\theta_j)\prod_{i=1}^{m-1}\frac{\sin(|\varepsilon-\sum_{j=1}^{i}\theta_j|(s_{i+1}-s_i))}{|\varepsilon-\sum_{j=1}^{i}\theta_j|}\hat{V}(\theta_i)d\theta_i\right) \tag{4.22}$$

which gives

$$\prod_{k=1}^{m}\int_0^{s_{k+1}} ds_k \int_{\mathbb{R}^d} d\varepsilon \frac{\sin(|\varepsilon|(t-s_m+s_1))}{|\varepsilon|}\left(\int_{\mathbb{R}^{d(m-1)}} \hat{V}(-\sum_{j=1}^{m-1}\theta_j) \prod_{i=1}^{m-1}\frac{\sin(|\varepsilon-\sum_{j=1}^{i}\theta_j|(s_{i+1}-s_i))}{|\varepsilon-\sum_{j=1}^{i}\theta_j|}\hat{V}(\theta_i)d\theta_i\right)$$

$\square$

4.5.3 Terms in the expansion for odd dimension

We want to show that the terms given by the formula in 4.5.2 correspond with the terms of the small t asymptotics of the wave trace. Ultimately we will do this with the wave to heat transform for odd dimension, but here we will calculate the first term for odd dimension, and show that the second term in the case of $d = 3$ contains only higher order t terms.

Let $m = 1$ and d be odd the the first term is given as

$$\begin{aligned}
\int_0^t ds \int_{\mathbb{R}^d} d\varepsilon \frac{\sin(|\varepsilon|t)}{|\varepsilon|}\widehat{V}(0) &= t\widehat{V}(0)\int_{\mathbb{R}^d} d\varepsilon \frac{\sin(|\varepsilon|t)}{|\varepsilon|} \\
&= \omega_{d-1}t\widehat{V}(0)\int_0^\infty dr\, r^{d-2}\sin(rt) \\
&= i2\omega_{d-1}t\widehat{V}(0)\int_0^\infty dr\, r^{d-2}(e^{irt}-e^{-irt}) \qquad (4.23)\\
&= i2\omega_{d-1}t\widehat{V}(0)\int_{-\infty}^\infty dr\, r^{d-2}e^{irt} \\
&= (-1)^{\frac{d-3}{2}}2\omega_{d-1}\widehat{V}(0)t\int_{-\infty}^\infty dr\, \frac{d^{d-2}}{dt^{d-2}}(e^{irt})
\end{aligned}$$

The above integral doesn't converge, so we must consider it as a tempered distribution. Take $\phi(t) \in \mathcal{S}(\mathbb{R})$ and consider the inner product.

$$\begin{aligned}\langle t\int\limits_{-\infty}^{\infty} dr\, \frac{d^{d-2}}{dt^{d-2}}(e^{irt}), \phi(t)\rangle &= \langle \delta^{(d-2)}(t), t\phi(t)\rangle \\ &= (t\phi(t))^{(d-2)}|_{t=0} \\ &= \phi^{(d-3)}(0)\end{aligned} \tag{4.24}$$

So we get (in the sense of distribution)

$$\begin{aligned}(-1)^{\frac{d-3}{2}}2\omega_{d-1}\widehat{V}(0)t\int\limits_{-\infty}^{\infty} dr\, \frac{d^{d-2}}{dt^{d-2}}(e^{irt}) &= (-1)^{\frac{d-3}{2}}2\omega_{d-1}\widehat{V}(0)\delta^{(d-3)}(t) \\ &= (-1)^{\frac{d-3}{2}}2\omega_{d-1}\left[\int\limits_{\mathbb{R}^d} V(x)\, dx\right]\delta^{(d-3)}(t)\end{aligned} \tag{4.25}$$

which is the first term in the small t asymptotics of the wave trace.

Next we will compute the second term in the $d = 3$ case so we may see it is of order $|t|$ and show the general strategy for computing the higher order terms. Let $m = 2$ and $d = 3$ the formula gives

$$\int\limits_0^t ds_2 \int\limits_0^{s_2} ds_1 \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} d\varepsilon d\theta\, \frac{\sin(|\varepsilon|(t-(s_2-s_1)))}{|\varepsilon|}\frac{\sin(|\varepsilon-\theta|(s_2-s_1))}{|\varepsilon-\theta|}\widehat{V}(\theta)\widehat{V}(-\theta) \tag{4.26}$$

We first observe that for $\psi(x) = -x$)

$$\widehat{V}(\theta)\widehat{V}(-\theta) = \int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3} e^{-i\theta(x-y)}V(x)V(y)\, dxdy = (V * V\circ\psi)\widehat{\ }(\theta). \tag{4.27}$$

Using this formula and a change of variables we can rewrite the integral as

$$\begin{aligned}\int\limits_0^t ds_2 \int_0^{s_2} ds_1 \int\limits_{\mathbb{R}^3} d\varepsilon \int\limits_{\mathbb{R}^3} d\theta (t-(s_2-s_1))(s_2-s_1)&\frac{\sin(|\varepsilon|(t-(s_2-s_1))}{|\varepsilon|(t-(s_2-s_1))} \\ &\frac{\sin(|\theta|(s_2-s_1))}{|\theta|(s_2-s_1)}(V * V\circ\psi)\widehat{\ }(\varepsilon+\theta)\end{aligned} \tag{4.28}$$

$$\widetilde{s}_2 = s_1 - s_2$$

$$\widetilde{s}_1 = s_1 + s_2$$

$$\int_0^t d\widetilde{s}_2 \int_{\widetilde{s}_2}^{2t-\widetilde{s}_2} d\widetilde{s}_1 \int_{\mathbb{R}^3} d\varepsilon \int_{\mathbb{R}^3} d\theta (t-\widetilde{s}_2)(\widetilde{s}_2) \frac{\sin(|\varepsilon|(t-\widetilde{s}_2)}{|\varepsilon|(t-\widetilde{s}_2)} \frac{\sin(|\theta|\widetilde{s}_2)}{|\theta|\widetilde{s}_2} (V * V \circ \psi)\widehat{\ }(\varepsilon+\theta) \tag{4.29}$$

Integrate w.r.t to $\widetilde{s}_1$

$$\int_0^t d\widetilde{s}_2 \int_{\mathbb{R}^3} d\varepsilon \int_{\mathbb{R}^3} d\theta \, 2(t-\widetilde{s}_2)^2(\widetilde{s}_2) \frac{\sin(|\varepsilon|(t-\widetilde{s}_2)}{|\varepsilon|(t-\widetilde{s}_2)} \frac{\sin(|\theta|\widetilde{s}_2)}{|\theta|\widetilde{s}_2} (V * V \circ \psi)\widehat{\ }(\varepsilon+\theta) \tag{4.30}$$

Now we represent the sinc function using the following formula with $a > 0$ and $\rho = |x|$.

$$\begin{aligned} \frac{1}{4\pi} \int_{S^2} e^{-iax\cdot y} d\sigma(y) &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} e^{-ia\rho\cos\theta} \sin\theta d\theta d\phi \\ &= \frac{1}{2} \int_0^{\pi} e^{-ia\rho\cos\theta} \sin\theta d\theta \\ &= \frac{1}{2} \int_{-1}^{1} e^{-ia\rho u} du \\ &= \frac{1}{-i2a\rho} \left. e^{-ia\rho u} \right|_{-1}^{1} \\ &= \frac{\sin a\rho}{a\rho} \end{aligned} \tag{4.31}$$

$$\begin{aligned} (4\pi)^2 \int_0^t d\widetilde{s}_2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \, 2(t-\widetilde{s}_2)^2(\widetilde{s}_2) & \int_{S^2} e^{-i(t-\widetilde{s}_2)\varepsilon\cdot\gamma_1} \, d\sigma(\gamma_1) \\ & \int_{S^2} e^{-i(\widetilde{s}_2)\theta\cdot\gamma_2} \, d\sigma(\gamma_2) (V * V \circ \psi)\widehat{\ }(\varepsilon+\theta) d\varepsilon d\theta \end{aligned} \tag{4.32}$$

expand the Fourier transform of $V * V$ and combine terms.

$$(4\pi)^2 \int_0^t d\widetilde{s}_2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} 2(t-\widetilde{s}_2)^2(\widetilde{s}_2) \int_{S^2} e^{-i\varepsilon\cdot(x+(t-\widetilde{s}_2)\gamma_1)}\, d\sigma(\gamma_1) \int_{S^2} e^{-i\theta\cdot(x+\widetilde{s}_2\gamma_2)}\, d\sigma(\gamma_2)(V*V)(-x)d\varepsilon d\theta dx \tag{4.33}$$

integrate over θ and ε

$$(4\pi)^2 \int_0^t d\widetilde{s}_2 \int_{\mathbb{R}^3} dx\, 2(t-\widetilde{s}_2)^2\widetilde{s}_2 \int_{S^2} \delta(x+(t-\widetilde{s}_2)\gamma_1)\, d\sigma(\gamma_1) \int_{S^2} \delta(x+\widetilde{s}_2\gamma_2)\, d\sigma(\gamma_2)(V*V)(-x) \tag{4.34}$$

integrate over x setting $x = -\widetilde{s}_2\gamma_2$

$$(4\pi)^2 \int_{S^2}\int_0^t d\widetilde{s}_2 \int_{S^2} 2(t-\widetilde{s}_2)^2(\widetilde{s}_2)\delta(-\widetilde{s}_2\gamma_2+(t-\widetilde{s}_2)\gamma_1)(V*V)(\widetilde{s}_2\gamma_2)\, d\sigma(\gamma_2)d\sigma(\gamma_1) \tag{4.35}$$

$$(4\pi)^2 \int_{S^2}\int_0^\infty d\widetilde{s}_2 \int_{S^2} \chi_{[0,t]}(\widetilde{s}_2)2(t-\widetilde{s}_2)^2(\widetilde{s}_2)\delta(-\widetilde{s}_2\gamma_2+(t-\widetilde{s}_2)\gamma_1) (V*V)(\widetilde{s}_2\gamma_2)\, d\sigma(\gamma_2)d\sigma(\gamma_1) \tag{4.36}$$

Integrate against a smooth function $\phi(t)$.

$$(4\pi)^2 \int_{\mathbb{R}} \phi(t) \int_{S^2}\int_0^\infty d\widetilde{s}_2 \int_{S^2} \chi_{[0,|t|]}(\widetilde{s}_2)2(|t|-\widetilde{s}_2)^2(\widetilde{s}_2)\delta(-\widetilde{s}_2\gamma_2+(|t|-\widetilde{s}_2)\gamma_1) (V*V)(\widetilde{s}_2\gamma_2)\, d\sigma(\gamma_2)d\sigma(\gamma_1)dt \tag{4.37}$$

We split the integral into to two pieces, $t > 0$ and $t < 0$.

$$I^+ = (4\pi)^2 \int_0^\infty \phi(t) \int_{S^2}\int_0^\infty d\widetilde{s}_2 \int_{S^2} \chi_{[0,t]}(\widetilde{s}_2)2(|t|-\widetilde{s}_2)^2(\widetilde{s}_2)\delta(-\widetilde{s}_2\gamma_2+(t-\widetilde{s}_2)\gamma_1) (V*V)(\widetilde{s}_2\gamma_2)\, d\sigma(\gamma_2)d\sigma(\gamma_1)dt \tag{4.38}$$

$$I^- = (4\pi)^2 \int_0^\infty \phi(-t) \int_{S^2} \int_0^\infty d\widetilde{s}_2 \int_{S^2} \chi_{[0,t]}(\widetilde{s}_2) 2(t-\widetilde{s}_2)^2(\widetilde{s}_2)\delta(-\widetilde{s}_2\gamma_2 + (t-\widetilde{s}_2)\gamma_1)$$
$$(V * V)(\widetilde{s}_2\gamma_2)\, d\sigma(\gamma_2) d\sigma(\gamma_1) dt \tag{4.39}$$

Consider I^+ and translate $\widetilde{t} = t - \widetilde{s}_2$

$$I^+ = (4\pi)^2 \int_{S^2} \int_0^\infty d\widetilde{s}_2 \int_{-\widetilde{s}_2}^\infty d\widetilde{t} \int_{S^2} \phi(\widetilde{t}+\widetilde{s}_2)\chi_{[0,\widetilde{t}+\widetilde{s}_2]}(\widetilde{s}_2) 2\widetilde{t}^2\, \widetilde{s}_2\, \delta(-\widetilde{s}_2\gamma_2 + \widetilde{t}\gamma_1)$$
$$(V * V)(\widetilde{s}_2\gamma_2)\, d\sigma(\gamma_2) d\sigma(\gamma_1) \tag{4.40}$$

Note that for $\widetilde{s}^2 \geq 0$, $\chi_{[0,\widetilde{t}+\widetilde{s}_2]}(\widetilde{s}_2) = 1$ exactly when $t \geq 0$.

$$I^+ = (4\pi)^2 \int_{S^2} \int_0^\infty d\widetilde{s}_2 \int_0^\infty d\widetilde{t} \int_{S^2} \phi(\widetilde{t}+\widetilde{s}_2) 2\widetilde{t}^2\, \widetilde{s}_2\, \delta(-\widetilde{s}_2\gamma_2 + \widetilde{t}\gamma_1)$$
$$(V * V)(\widetilde{s}_2\gamma_2)\, d\sigma(\gamma_2) d\sigma(\gamma_1) \tag{4.41}$$

integrate over $\widetilde{t}^2 d\sigma(\gamma_1) d\widetilde{t}$ which sets $\widetilde{s}_2\gamma_2 = \widetilde{t}\gamma_1$

$$I^+ = (4\pi)^2 \int_0^\infty d\widetilde{s}_2\, 2\, \widetilde{s}_2 \phi(2\widetilde{s}_2) \int_{S^2} (V * V)(\widetilde{s}_2\gamma_2)\, d\sigma(\gamma_2) \tag{4.42}$$

We now consider the negative portion of the integral, I^-

$$I^- = (4\pi)^2 \int_0^\infty \phi(-t) \int_{S^2} \int_0^\infty d\widetilde{s}_2 \int_{S^2} \chi_{[0,t]}(\widetilde{s}_2) 2(t-\widetilde{s}_2)^2(\widetilde{s}_2)\delta(-\widetilde{s}_2\gamma_2 + (t-\widetilde{s}_2)\gamma_1)$$
$$(V * V)(\widetilde{s}_2\gamma_2)\, d\sigma(\gamma_2) d\sigma(\gamma_1) dt \tag{4.43}$$

We again translate and set $\widetilde{t} = t - \widetilde{s}_2$

$$I^- = (4\pi)^2 \int_{S^2} \int_0^\infty d\widetilde{s}_2 \int_{-\widetilde{s}_2}^\infty d\widetilde{t} \int_{S^2} \phi(-(\widetilde{t}+\widetilde{s}_2))\chi_{[0,\widetilde{t}+\widetilde{s}_2]} 2\widetilde{t}^2\, \widetilde{s}_2\, \delta(-\widetilde{s}_2\gamma_2 - \widetilde{t}\gamma_1)$$
$$(V * V)(\widetilde{s}_2\gamma_2)\, d\sigma(\gamma_2) d\sigma(\gamma_1) \tag{4.44}$$

The characteristic functions behaves as in the I^+ case giving

$$I^- = (4\pi)^2 \int_{S^2} \int_0^\infty d\widetilde{s}_2 \int_0^\infty d\widetilde{t} \int_{S^2} \phi(-(\widetilde{t}+\widetilde{s}_2)) 2\widetilde{t}^2\, \widetilde{s}_2\, \delta(-\widetilde{s}_2\gamma_2 - \widetilde{t}\gamma_1) \\ (V*V)(\widetilde{s}_2\gamma_2)\, d\sigma(\gamma_2) d\sigma(\gamma_1) \tag{4.45}$$

integrate over $\widetilde{t}^2 d\sigma(\gamma_1) d\widetilde{t}$ which sets $\widetilde{s}_2\gamma_2 = \widetilde{t}\gamma_1$

$$I^- = (4\pi)^2 \int_0^\infty d\widetilde{s}_2\, \phi(-2\widetilde{s}_2) 2\, \widetilde{s}_2 \int_{S^2} (V*V)(\widetilde{s}_2\gamma_2)\, d\sigma(\gamma_2) \tag{4.46}$$

Adding the positive and negative pieces we have

$$I^+ + I^- = (4\pi)^2 \int_{-\infty}^\infty d\widetilde{s}_2\, \phi(2\widetilde{s}_2) 2\, |\widetilde{s}_2| \int_{S^2} (V*V)(\widetilde{s}_2\gamma_2)\, d\sigma(\gamma_2) \tag{4.47}$$

rescaling gives

$$(4\pi)^3 \int_{-\infty}^\infty d\widetilde{s}_2\, \phi(\widetilde{s}_2) |\widetilde{s}_2| \fint_{S^2} (V*V)(\frac{\widetilde{s}_2}{2}\gamma_2)\, d\sigma(\gamma_2) \tag{4.48}$$

So in the sense of distribution we have

$$(4\pi)^3 |\widetilde{s}_2| \fint_{S^2} (V*V)(\frac{\widetilde{s}_2}{2}\gamma_2)\, d\sigma(\gamma_2) \sim 4(4\pi)^3 |\widetilde{s}_2| (V*V)(0) = (4\pi)^3 |\widetilde{s}_2|\, \|V\|_{L^2}^2 \tag{4.49}$$

as $\widetilde{s}_2 \to 0$. Thus our second term is given by $(4\pi)^3|t|\|V\|_2^2$.

4.5.4 Wave to Heat Transform

We have that for odd dimension $Tr(W_V(t) - W_0(t)) \sim \sum_{j=1}^{\frac{d-1}{2}} w_j + \sum_{j=\frac{d+1}{2}}^{\infty} w_j |t|^{2j-d}$. The next step is to show that the w_j's are multiples of the heat invaraints (as described in Proposition 3.1.2) associated to $-\Delta + V$. This will be shown using the wave to heat transform, which gives the heat invariants in terms of the wave invariants.

Proposition 4.5.3. *Let a_j be the heat invariants as described in Lemma 3.1.1, and w_j be the wave invariants. Then the following formula holds for odd dimensions.*

$$w_j = \begin{cases} \frac{2^{2(j-d)+1}}{M_j} a_j & 1 \le j \le \frac{d-1}{2} \\ \frac{2^{2(j-d)+1}}{N_j} a_j & j \ge \frac{d+1}{2} \end{cases} \tag{4.50}$$

Where M_j and N_j are non-zero constants given by the formulas

$$M_j = \frac{(-1)^{\frac{d-1-2j}{2}}}{(2\pi)^{\frac{1}{2}}} \int \varepsilon^{d-1-2j} e^{-\varepsilon^2} \, d\varepsilon$$

$$N_j = \int e^{-\theta^2} |\theta|^{2j-d} d\theta \tag{4.51}$$

We begin by considering the formula

$$e^{-tx^2} = \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{\frac{-s^2}{4t}} \cos(sx) \, ds \tag{4.52}$$

and applying it to the regularized heat trace. We then observe that this is an integral of the kernel against the wave trace.

$$\begin{aligned}
Tr(e^{-tH_V} - e^{-tH_0}) &= Tr \left[(4\pi t)^{-\frac{1}{2}} \int e^{-\frac{s^2}{4t}} \left[\cos(s\sqrt{(-\Delta + V)}) \right.\right. \\
&\qquad \left.\left. - \cos(s\sqrt{(-\Delta)}) \right] ds \right] \\
&= (4\pi t)^{-\frac{1}{2}} \int e^{-\frac{s^2}{4t}} Tr \left[\cos(s\sqrt{(-\Delta + V)}) \right. \\
&\qquad \left. - \cos(s\sqrt{(-\Delta)}) \right] ds \\
&= \frac{1}{2} (4\pi t)^{-\frac{1}{2}} \int e^{-\frac{s^2}{4t}} Tr(W(s) - W_0(s)) ds \\
&= \frac{1}{2} (4\pi t)^{-\frac{1}{2}} \int e^{-\frac{s^2}{4t}} \left[\sum_{j=1}^{\frac{d-1}{2}} w_j \delta^{d-1-2j}(s) \right. \\
&\qquad \left. + \sum_{j \geq \frac{d+1}{2}} w_j |s|^{2j-d} \right] ds.
\end{aligned} \tag{4.53}$$

We now have two model integral to solve. We will begin by fixing j with $1 \leq j \leq \frac{d-1}{2}$.

$$\begin{aligned}
I_j &= \frac{1}{2}\frac{1}{\sqrt{4t}}w_j \int e^{-\frac{s^2}{4t}}\delta^{d-1-2j}(s)ds \\
&= \frac{1}{2}\frac{1}{\sqrt{4t}}w_j\,(-1)^{d-1-2j}\int \delta(s)\left(\frac{d}{ds}\right)^{d-1-2j} e^{-\frac{s^2}{4t}}ds \\
&= \frac{1}{2}\frac{1}{\sqrt{4t}}w_j \int \delta(\sqrt{4t}\theta)\left(\frac{1}{\sqrt{4t}}\right)^{d-1-2j}\left(\frac{d}{d\theta}\right)^{d-1-2j} e^{-\theta^2}\sqrt{4t}d\theta \qquad (4.54) \\
&= \frac{1}{2}\left(\frac{1}{\sqrt{4t}}\right)^{d-2j} w_j \int \delta(\theta)\left(\frac{d}{d\theta}\right)^{d-1-2j} e^{-\theta^2}d\theta \\
&= \frac{1}{2}\,(4t)^{-\frac{d}{2}}\,w_j\,(4t)^j\left[\left(\frac{d}{d\theta}\right)^{d-1-2j} e^{-\theta^2}\right]_{\theta=0}.
\end{aligned}$$

using the substitution $\theta = \frac{s}{\sqrt{4t}}$ and note that $\frac{d}{ds} = \frac{1}{\sqrt{4t}}\frac{d}{d\theta}$

For $j \geq \frac{d+1}{2}$ we have the following integral.

$$\begin{aligned}
J_j &= \frac{1}{2}\frac{1}{\sqrt{4t}}w_j \int e^{-\frac{s^2}{4t}}|s|^{2j-d}ds \\
&= \frac{1}{2}\frac{1}{\sqrt{4t}}w_j \int e^{-\theta^2}|\sqrt{4t}\theta|^{2j-d}\sqrt{4t}d\theta \\
&= \frac{1}{2}(\sqrt{4t})^{2j-d}w_j \int e^{-\theta^2}|\theta|^{2j-d}d\theta \\
&= \frac{1}{2}(4t)^{-\frac{d}{2}}w_j(4t)^j \int e^{-\theta^2}|\theta|^{2j-d}d\theta
\end{aligned} \qquad (4.55)$$

Comparing coefficients of the powers of t we get a formula for the wave trace coefficients in terms of the heat trace coefficients. Fixing d and defining M_j and N_j

as

$$
\begin{aligned}
M_j &= \left[\left(\frac{d}{d\theta}\right)^{d-1-2j} e^{-\theta^2}\right]_{\theta=0} \\
N_j &= \int e^{-\theta^2}|\theta|^{2j-d} d\theta
\end{aligned}
\tag{4.56}
$$

N_j is clearly always positive. We can see that for odd dimension M_j is positive and non-zero by looking at the Fourier transform of $e^{-\theta^2}$

$$
\begin{aligned}
\left[\left(\frac{d}{d\theta}\right)^{d-1-2j} e^{-\theta^2}\right]_{\theta=0} &= \left[\left(\frac{d}{d\theta}\right)^{d-1-2j} \frac{1}{(2\pi)^{\frac{1}{2}}} \int e^{-\varepsilon^2} e^{i\varepsilon\theta}\, d\varepsilon\right]_{\theta=0} \\
&= \left[\frac{(-1)^{\frac{d-1-2j}{2}}}{(2\pi)^{\frac{1}{2}}} \int \varepsilon^{d-1-2j} e^{-\varepsilon^2} e^{i\varepsilon\theta}\, d\varepsilon\right]_{\theta=0} \\
&= \frac{(-1)^{\frac{d-1-2j}{2}}}{(2\pi)^{\frac{1}{2}}} \int \varepsilon^{d-1-2j} e^{-\varepsilon^2}\, d\varepsilon
\end{aligned}
\tag{4.57}
$$

For odd dimension we get that $M_j \neq 0$. This then gives the following formula for the wave coefficients, w_j.

$$
w_j = \begin{cases} \frac{2(4)^{j-d}}{M_j} a_j & 1 \leq j \leq \frac{d-1}{2} \\ \frac{2(4)^{j-d}}{N_j} a_j & j \geq \frac{d+1}{2} \end{cases}
\tag{4.58}
$$

4.6 The Poisson Formula

The relation of the resonances to the trace of the wave group is the connection that will allow us use the iso-resonant condition to prove compactness. The Poisson Formula is what provides this connection.

Proposition 4.6.1. *(Poisson Formula) [6]*
Let $V \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$ with d odd. Then

$$Tr(W_V(t) - W_0(t)) = \sum_{\lambda \in Res(V)} m_\lambda e^{i\lambda|t|},\ t \neq 0 \tag{4.59}$$

where m_λ is the multiplicity of the resonance and $k \geq d$ in the sense of distribution.

The proposition follows from using along with the Birman-Krešin formula and Hadamard factorization. An outline of the proof will be given here, for a full proof see Dyatlov-Zworksi [8].

Lemma 4.6.1. *[8] Suppose $V \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$ is real valued with d odd.*
Then for $f \in \mathcal{S}(R)$ the operator $f(H_V) - f(H_0)$ is trace class and

$$Tr(f(H_V) - f(H_0)) = \frac{1}{2\pi i}\int_0^\infty f(\lambda^2) tr(S(\lambda)^{-1}\partial_\lambda S(\lambda)) d\lambda + \sum_{k=1}^{K} f(E_k) + \frac{1}{2}\widetilde{m}(0) f(0), \tag{4.60}$$

where $S(\lambda)$ is the scattering matrix and $E_k < 0$ are eigenvalues of H_V.

Using the representation

$$Tr\left(W_V(t) - W_0(t)\right) = 2Tr\left[\cos\left(t\sqrt{H_V}\right) - \cos\left(t\sqrt{H_0}\right)\right] \tag{4.61}$$

we note that $\cos(\lambda)$ is not in $\mathcal{S}(\mathbb{R})$. However, if we consider the formula $\cos(\lambda) = \frac{e^{i\lambda}+e^{-i\lambda}}{2}$ for $\rho(t) \in C_0^\infty(\mathbb{R})$ the Fourier transform $\widehat{\rho}(s)$ then we see that

$$\widehat{\rho}(\sqrt{H_V}) = \int \rho(t) e^{-it\sqrt{H_V}}\, dt. \tag{4.62}$$

The function $\widehat{\rho}(s) \in \mathcal{S}(\mathbb{R})$ so we can take $f(\lambda) = \widehat{\rho}(\sqrt{\lambda}) + \widehat{\rho}(-\sqrt{\lambda})$ in the Birman-Krein formula. Then in a sense of distribution we get that the trace of $W_V(t) - W_0(t)$ exists and has the expansion given in the lemma. Dyatlov and Zworski then use the relation $tr(S(\lambda)^{-1}\partial_\lambda S(\lambda)) = \partial_\lambda(\log \det S(\lambda))$ and a factorization of $\partial_\lambda(\log \det S(\lambda))$ to obtain the result.

Chapter 5

To establish the compactness result of Theorem 1.0.2 we will adapt the tools from Brüning and Donnelly presented in chapter 2. The uniform bounds on the Sobolev semi-norms of the potentials in $\mathrm{IsoRes}(V_0)$ will allow us to show convergence of an isoresonant sequence $\{V_i\} \subset \mathrm{IsoRes}(V_0)$ in a Fréchet metric induced by the semi-norms. Finally, we will show that limiting funcition is in the isoresonant set.

5.1 Uniform bounds on $W_{k,2}$ norms (Brüning/Donnelly Mechanics)

We can write down a formula for the wave invariants using the formula for the heat invariants given in $\mathbb{R}^d$ from 3.1.2 along with the relation between the heat and wave invariants from 4.5.3. We begin by restating the definition of the set, $\mathcal{A}_{j,k}$, from chapter 3 of k-tuples of multi-indicies that we will use in the formula for the wave invariants.

$$\mathcal{A}_{j,k} = \left\{ \alpha = (\alpha^1, \cdots, \alpha^k) \;\middle|\; \begin{array}{c} \alpha^i \in \mathbb{N}_0^d \text{ for } 1 \le i \le k \\ |\alpha^i| \le j-k, \\ \sum_{i=1}^k |\alpha^i| = 2(j-k) \\ \sum_{i=1}^k \alpha^i_l \text{ is even for each } l. \end{array} \right\}$$

The jth coefficient,w_j, of the small t asymptotics is independent of our choice of $V \in \mathrm{IsoRes}(V_0)$ and is given by the following formula:

$$w_j = d_j \int |\nabla^{j-2} V|^2 + \sum_{k=3}^{j} \sum_{\alpha \in \mathcal{A}_{j,k}} d_\alpha \int D^{\alpha^1}(V) D^{\alpha^2}(V) \cdots D^{\alpha^k}(V). \tag{5.1}$$

A rearrangement gives that for each j we have the bound

$$\|V\|^2_{j-2,2} \le C \left(1 + \sum_{k=3}^{j} \sum_{\alpha \in \mathcal{A}_{j,k}} \int |D^{\alpha^1}(V) D^{\alpha^2}(V) \cdots D^{\alpha^k}(V)| \right) \tag{5.2}$$

where $C > 0$ is independent of our choice of V from the isoresonant set. The strategy is to then show each term

$$\int |D^{\alpha^1}(V) D^{\alpha^2}(V) \cdots D^{\alpha^k}(V)|$$

is bounded by a constant independent of our choice of V ($d = 1$) or by a multiple of $1 + \|V\|^{\beta}_{j-2,2}$ with $\beta < 2$ ($d \ge 3$). Together these bounds yield a uniform bound on $\|V\|_{j-2,2}$ for each j.

Lemma 5.1.1. *Let $u \in C_0^1(\mathbb{R})$ then*

$$\|u\|_\infty \le C \|u\|_{1,2}$$

Proof. This follows from Theorem 6.0.3 with $d = 1$.

□

Proposition 5.1.1. *Let $d = 1$ and $\|V\|_{j-3,2} \le M$, $V \in \mathrm{IsoRes}(V_0)$ then for $j \ge 3$*

$$\int |D^{\alpha^1}(V)D^{\alpha^2}(V)\cdots D^{\alpha^k}(V)| \le C$$

where C depends only on M and j.

Proof: We use the bounds on the order of the $D^{\alpha^i}V$ terms to conclude that each term has order less that $j-3$ as $|\alpha^i| \le j-k \le j-3$ and there are at most 2 terms with order $j-3$ as $\sum_{i=1}^{k} |\alpha^i| \le 2(j-k) \le 2(j-3)$. Lemma 5.1.1 will then allow us to get the desired bounds.

Case 1: Assume 0 terms of order $j-3$

Then using the Lemma for each i

$$|D^{\alpha^i}(V)| \le C\|D^{\alpha^i}(V)\|_{1,2} \le C\|V\|_{j-3,2}$$

This gives

$$\int |D^{\alpha^1}(V)D^{\alpha^2}(V)\cdots D^{\alpha^k}(V)| \le C^k M^k \le C^j M^j$$

(WLOG we may assume $C, M \ge max(1, m(B_r))$)

Case 2: Assume 1 term is of order $j-3$

Using the results from Case 1 with the Hölder inequality we have

$$\begin{aligned}\int |D^{\alpha^1}(V)D^{\alpha^2}(V)\cdots D^{\alpha^k}(V)| &\le C^{k-1}M^{k-1}\int |D^{\alpha^1}(V)| \\ &\le C^{k-1}M^{k-1}m(B_r)\|V\|_{j-3,2} \\ &\le C^j M^j\end{aligned} \tag{5.3}$$

Case 3: Assume 2 terms of order $j-3$

Again using the results of Case 1 and the Hölder inequality gives

$$\begin{aligned}\int |D^{\alpha^1}(V)D^{\alpha^2}(V)\cdots D^{\alpha^k}(V)| &\le C^{k-2}M^{k-2}\int |D^{\alpha^1}(V)D^{\alpha^2}(V)| \\ &\le C^{k-2}M^{k-2}\|V\|_{j-3,2}^2 \\ &\le C^j M^j\end{aligned} \tag{5.4}$$

□

For $d \ge 3$ we use a proof given by Donnelly which requires reordering the $D^{\alpha^i}(V)$ terms. Fixing k, we reorder the terms according to $|\alpha^i|$ and define T in the following way

$$T = D^{\alpha^1}(V)D^{\alpha^2}(V)\cdots D^{\alpha^l}(V)D^{\alpha^{l+1}}(V)\cdots D^{\alpha^k}(V)$$

where the ordering is chosen s.t.

$$\begin{aligned} i \le l &\Rightarrow d > 2(j - |\alpha^i| - 3) \\ i > l &\Rightarrow d \le 2(j - |\alpha^i| - 3) \end{aligned} \tag{5.5}$$

We will separate $\int |T|$ using the generalized Hölder's inequality and apply the Sobolev embedding theorem to get an estimate. Note that the conditions determine which case of the Sobolev embedding theorem for $p = 2$ and $k = (j - |\alpha^i| - 3)$ is appropriate.

Proposition 5.1.2. *(Lemma 4.6, Donnelly) If $d \ge 3, j > \frac{n}{2} + 1$, and $\|V\|_{j-3,2} \le C_1$, then*

$$\int |T| \le C_2 \left(1 + \|V\|^{\beta}_{j-2,2}\right)$$

where $\beta < 2$ and C_2 depends on C_1

Proof. We will look at the possible values of l and for each case the general strategy will be to use the generalized Hölder's inequality to show

$$\int |T| \le C \prod_{i=1}^{k} \|D^{\alpha^i}(V)\|_{r_i}$$

with $\sum\limits_{i=1}^{k} \frac{1}{r_i} = 1$.

For $i > l$ we use two Sobolev inequalities

$$\|D^{\alpha^i}(V)\|_{\infty} \le C\|V\|_{j-3,2}$$

when $d < 2(j - |\alpha^i| - 3)$ and

$$\|D^{\alpha^i}(V)\|_{r_i} \le C\|V\|_{j-3,2}$$

where $2 \le r_i < \infty$ when $d = 2(j - |\alpha^i| - 3)$. These two inequalities give the bound

$$\int |T| \le C \prod_{i=1}^{k} \|D^{\alpha^i}(V)\|_{r_i} \le \tilde{C}\|V\|^{k-l}_{j-3,2} \prod_{i=1}^{l} \|D^{\alpha^i}(V)\|_{r_i}$$

The remainder of the proof is to show that for $1 \le i \le l$ we can choose r_i to apply the appropriate Sobolev inequality.

Case 0: When $l = 0$ the estimate holds for $\beta = 0$ using the above method.
Case 1: Letting $l = 1$ implies $d > 2(j - |\alpha^1| - 3)$, so setting

$$r_1 = \frac{2d}{d - 2(j - |\alpha^1| - 3)}$$

yields

$$\|D^{\alpha^1}(V)\|_{r_1} \leq C\|D^{\alpha^i}(V)\|_{j-|\alpha^1|-3,2} \leq C\|V\|_{j-3,2}$$

by the generalized Sobolev inequality. The only condition on j is $2 \leq \frac{2d}{d-2(j-|\alpha^1|-3)}$ or $j-3 \geq |\alpha^1|$ which is true for every $D^{\alpha^i}(V)$ and j.

Since $\frac{1}{r_1} \leq \frac{1}{2}$, we can choose the remaining r_i's to meet the condition $\sum_{i=1}^{k} \frac{1}{r_i} = 1$. So for $l = 1$ we have the bound with $\beta = 0$.

Case 2: Assume $l = 2$.

If $|\alpha^1|$ and $|\alpha^2|$ are such that r_1 and r_2 (as chosen in case $l = 1$) satisfy

$$\frac{1}{r_1} + \frac{1}{r_2} < 1$$

then we proceed as in the case $l = 1$ and apply the generalized Hölder's inequality to get the result with $\beta = 0$.

Now assume $\frac{1}{r_1} + \frac{1}{r_2} = 1$. Since $|\alpha^i| \leq j-3$, this implies $|\alpha^1| = |\alpha^2| = j-3$ and thus $r_1 = r_2 = 2$. We may then apply the generalized Hölder's inequality to get.

$$\int |T| \leq C\|D^{\alpha^1}(V)\|_{r_1+\varepsilon} \prod_{i=2}^{k} \|D^{\alpha^i}(V)\|_{r_i}$$

Where $\varepsilon > 0$ and r_i for $i > 3$ are chosen to satisfy the Hölder condition. Furthermore if we choose ε such that $r_1 + \varepsilon < \frac{2d}{d-2}$ then the general Sobolev inequality gives that

$$\|D^{\alpha^1}(V)\|_{r_1+\varepsilon_i} \leq C_1\|D^{\alpha^1}(V)\|_{1,2} \leq C_2\|V\|_{j-2,2}$$

So we get the result with $\beta = 1$.

Case 3: Assume $l \geq 3$ and $d > 2(j-|\alpha^i|-2)$ for $i = 1, 2$

Let r_i be as in case 1 and 2 and set

$$s_i = \frac{2d}{d-2(j-|\alpha^i|-2)}$$

L_p interpolation gives that for $0 < \varepsilon_i < 1$ there exists a $0 < \beta_i < 1$ s.t.

$$\|D^{\alpha^i}(V)\|_{r_i+\varepsilon_i} \leq \|D^{\alpha^i}(V)\|_{r_i}^{\beta_i}\|D^{\alpha^i}(V)\|_{s_i}^{1-\beta_i}$$

Thus using the generalized Hölder's inequality we have

$$\begin{aligned}
\int |T| &\leq C\|D^{\alpha^1}(V)\|_{r_1+\varepsilon_1}\|D^{\alpha^2}(V)\|_{r_2+\varepsilon_2}\prod_{i=3}^{k}\|D^{\alpha^i}(V)\|_{r_i} \\
&\leq \|D^{\alpha^1}(V)\|_{r_1}^{\beta_1}\|D^{\alpha^1}(V)\|_{s_1}^{1-\beta_1}\|D^{\alpha^2}(V)\|_{r_2}^{\beta_2}\|D^{\alpha^2}(V)\|_{s_2}^{1-\beta_2}\prod_{i=3}^{k}\|D^{\alpha^i}(V)\|_{j-|\alpha^i|-3,2} \\
&\leq C\|V\|_{j-3,2}^{\beta_1}\|D^{\alpha^1}(V)\|_{s_1}^{1-\beta_1}\|V\|_{j-3}^{\beta_1}\|D^{\alpha^2}(V)\|_{s_2}^{1-\beta_2}\prod_{i=3}^{k}\|V\|_{j-3,2} \\
&\leq C\|D^{\alpha^1}(V)\|_{s_1}^{1-\beta_1}\|D^{\alpha^2}(V)\|_{s_2}^{1-\beta_2} \\
&\leq C\|V\|_{j-2,2}^{\beta}
\end{aligned} \tag{5.6}$$

Where $\beta < 2$. r_i may be chosen arbitrarily for $i > l$ and as in case 1 for $i \leq l$, so in order to satisfy the Hölder condition we require.

$$\frac{1}{r_1+\varepsilon_1}+\frac{1}{r_2+\varepsilon_2}+\sum_{i=3}^{l}\frac{1}{r_i}<1$$

Which, for sufficiently large $\varepsilon_1, \varepsilon_2 < 1$ is implied by

$$\frac{1}{s_1}+\frac{1}{s_2}+\sum_{i=3}^{l}\frac{1}{r_i}<1$$

Substituting for s_i and r_i gives

$$\sum_{i=1}^{2}\frac{d-2(j-|\alpha^i|-2)}{2d}+\sum_{i=3}^{l}\frac{d-2(j-|\alpha^i|-3)}{2d}<1$$

which may be rewritten as

$$(d-2j-6)l+2\sum_{i=1}^{l}|\alpha^i|<2d+4$$

Because $\sum_{i=1}^{l}|\alpha^i| \leq 2(j-k)$ it is sufficient to show

$$(d-2j-6)l+4(j-k)<2d+4$$

Using assumption $l \geq 3$ lets us rewrite the inequality as

$$\frac{d}{2} + 3 - \frac{2k-4}{l-2} < j$$

Then $k \geq l \geq 3$ gives $\frac{2k-4}{l-2} \geq \frac{2k-4}{k-2} = 2$ so it suffices for

$$\frac{d}{2} + 1 < j$$

Case 4: $l \geq 3$ and $d \leq 2(j - |\alpha^i| - 2)$ for $D^{\alpha^1}(V)$ and $D^{\alpha^2}(V)$.

For $2 \leq s < \infty$ we have the embedding

$$\|D^{\alpha^i}(V)\|_s \leq C\|D^{\alpha^i}(V)\|_{j-|\alpha^i|-2,2} \leq C\|V\|_{j-2,2}.$$

Then L_p interpolation gives for $2 < t < s$

$$\|D^{\alpha^i}(V)\|_t \leq \|D^{\alpha^i}(V)\|_s^{\beta_i}\|D^{\alpha^i}(V)\|_2^{1-\beta_i}.$$

We may take t to be arbitrarily large reducing the Hölder condition to

$$\sum_{i=3}^{l} \frac{1}{r_i} < 1.$$

If $l = 3$ the condition is met as $r_3 \geq 2$, so we assume $l \geq 4$. Substituting for r_i and rewriting the inequality we get

$$(l-2)(d-2j+6) + 2\sum_{i=3}^{l} |\alpha^i| < 2d.$$

Using the inequality $\sum_{i=3}^{l} |\alpha^i| \leq \sum_{i=1}^{k} |\alpha^i| \leq 2(j-k)$ gives the sufficient condition

$$(l-2)(d-2j+6) + 4(j-k) < 2d$$

which can be recast as

$$(l-4)d + 6(l-2) - 4k < (2l-8)j.$$

If $l = 4$ then the inequality reduces to $12 - 4k < 0$ which always holds as $4 = l \leq k$. For $l \geq 5$ we rewrite the inequality as

$$\frac{d}{2} + 3\frac{l-4}{l-4} + 3\frac{2}{l-4} - \frac{2k}{l-4} < j$$

which reduces to

$$\frac{d}{2} + 3 - \frac{(2k-6)}{l-4} < j.$$

Since $k \geq l$ it is sufficient for the following series of inequalities to hold:

$$\frac{d}{2} + 3 - \frac{(2k-6)}{k-4} < j$$

$$\frac{d}{2} + 1 - \frac{(2)}{k-4} < j \tag{5.7}$$

$$\frac{d}{2} + 1 < j.$$

This gives the condition on j.

$\square$

Definition 5.1.1. *Let $\mathcal{V}_{s,r,d}$ be the set of potentials isoresonant to a fixed potential V_0 whose support is containted in a ball of raidus r and with uniform bounds on the $W^{s,2}$ norms.*

$$\mathcal{V}_{s,r,d} = \{V \in \mathrm{IsoRes}(V_0) \subset C_0^\infty(\mathbb{R}^d, \mathbb{R}) |\ \mathit{supp}\ V \subset B_r(0), \|V\|_{s,2} < C\}$$

with V_0 fixed and $C, s, r, d > 0$. Furthermore let $\mathcal{V}_{r,d}$ be

$$\mathcal{V}_{r,d} = \{V \in C_0^\infty(\mathbb{R}^d, \mathbb{R}) |\ \mathrm{supp}\ V \subset B_r(0), \|D^\alpha V\|_{L^2} < C_\alpha\, \forall \alpha\}$$

Theorem 5.1.1. *(Donnelly 4.1) If $s > \frac{d}{2} - 2$ then $\mathcal{V}_{s,r,d} \subset \mathcal{V}_{r,d}$.*

Suppose $s > \frac{d}{2} - 2$. Then there is a uniform bound on $j-3$ Sobolev norm when $s = j - 3$. Furthermore, this implies $j - 3 > \frac{d}{2} - 2$ or $j > \frac{d}{2} + 1$, so by proposition 5.1.2 there is a uniform bound on the $j - 2 = s + 1$ Sobolev norm. Induction then gives that for each $t < \infty$ there is a uniform bound on $\|V\|_{t,2}$ for $V \in \mathcal{V}_{s,r,d}$.

Theorem 5.1.2. *For $d \leq 3$, $\mathrm{IsoRes}(V_0, r, d) \subset \mathcal{V}_{r,d}$*

By Theorem 5.1.1, for $d \leq 3$ we only need an apriori bound on $\|V\|_{L^2}$, which we get from the second term in the expansion of the wave trace.

5.2 Compactness in the Fréchet Space

To show that our isoresonant set is compact we will need to define the Fréchet metric for which the compactness applies. Next we will show that the isoresonant set lies inside a compact set and is closed.

Definition 5.2.1. *Let $V, W \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$ and let $\{\alpha^i\}$ enumerate the set $\{\alpha\}$ of all d-length multi-indexes. Then for this enumeration we define a Fréchet metric*

$$\rho_F(V, W) = \sum_i 2^{-i} \frac{\|D^{\alpha^i}(V - W)\|_{L^2}}{1 + \|D^{\alpha^i}(V - W)\|_{L^2}}$$

Definition 5.2.2. *Let $\mathcal{V}_{r,d}$ be the set*

$$\mathcal{V}_{r,d} = \{V \in C_0^\infty(\mathbb{R}^d, \mathbb{R}) |\ supp\ V \subset B_r(0), \|V\|_{W_{j,2}} < C_j\ , \forall j\}$$

Lemma 5.2.1. *$\mathcal{V}_{r,d}$ is equicontinuous in every derivative.*

Proof. Let $x, y \in B_r(0)$, α be a d length multi-index, and $V \in \mathcal{V}_{r,d}$. To show $\mathcal{V}_{r,d}$ is equicontinuous in each derivative it is enough to show

$$|D^\alpha V(x) - D^\alpha V(y)| \leq C_\alpha |x - y|$$

where C_α depends only on α.

From the Appendix we have Lemma Theorem 6.0.3 which gives

$$\|V\|_{L^\infty} \leq \|V\|_{W_{d_0,2}} \text{ for } d_0 > \frac{d}{2}$$

Thus

$$\|D^\alpha V\|_{L^\infty} \leq \|V\|_{W_{|\alpha|+d_0,2}} \text{ for } d_0 > \frac{d}{2}$$

Since for all $V \in \mathcal{V}_{r,d}$ the $W_{j,2}$ norms have a uniform bound with respect to j, then we get a uniform bound on the L^∞ norms of $D^\alpha V$ Which depends only on $|\alpha|$.

Now fix α and consider $x, y \in B_r(0)$. Then by the mean value theorem 6.0.2

$$|D^\alpha V(x) - D^\alpha V(y)| \leq \|\nabla(D^\alpha V)\|_{L^\infty} |x - y| \leq C_{|\alpha|+1+d_0} |x - y|$$

Where C is uniform with respect to $\mathcal{V}_{r,d}$, thus $\mathcal{V}_{r,d}$ is equicontinuous in the α^{th} derivative for each α.

□

Proposition 5.2.1. *$\mathcal{V}_{r,d}$ is compact with respect to the Fréchet metric.*

Proof. Let $\{V_n\} \subset \mathcal{V}_{r,d}$

Lemma 5.2.1 gives $\{V_n\}$ is equicontinuous , so there exists a uniformly convergent subsequence $\{V_{n_k}\}$. This new sequences is equicontinuous in each derivative so we take a sequence of convergent sub sequences $\{V_{n_{k_i}}^{\alpha^i}\}$ for each i.

Diagonalizing the sequence of sequences $\{V_{n_{k_i}}\}$ yields the subsequence $\{V_{n_j}\}$ such that

$V_{n_j} \to V$ uniformly in each derivative pointwise.

The support of each V_n is contained in $B_r(0)$, so

$$V_{n_j} \to V \text{ in each semi-norm}$$

as we can choose J (for a fixed i) such that $j > J$ implies

$$\|D^{\alpha^i} V_{n_j} - D^{\alpha^i} V\|_{L^2} < \varepsilon Vol(B_r(0)).$$

We claim the $V_{n_j} \to V$ in the C^∞ metric. Consider

$$\|V_{n_j} - V\|_F = \sum_i 2^{-i} \frac{\|D^{\alpha^i} V_{n_j} - D^{\alpha^i} V\|_{L^2}}{1 + \|D^{\alpha^i} V_{n_j} - D^{\alpha^i} V\|_{L^2}}.$$

Choose l such that

$$\sum_{i=l+1}^{\infty} 2^{-i} \frac{\|D^{\alpha^i} V_{n_j} - D^{\alpha^i} V\|_{L^2}}{1 + \|D^{\alpha^i} V_{n_j} - D^{\alpha^i} V\|_{L^2}} < \frac{\varepsilon}{2}.$$

We then choose J such that $j > J$ and $i \leq l$ implies

$$\|D^{\alpha^i} V_{n_j} - D^{\alpha^i} V\|_{L^2} < \frac{\varepsilon 2^{i-1}}{l + 1 - \varepsilon 2^{i-1}}.$$

Then for $j > J$

$$\sum_{i=0}^{l} 2^{-i} \frac{\|D^{\alpha^i} V_{n_j} - D^{\alpha^i} V\|_{L^2}}{1 + \|D^{\alpha^i} V_{n_j} - D^{\alpha^i} V\|_{L^2}} < \frac{\varepsilon}{2}.$$

Thus $V_{n_j} \to V$ in the C^∞ norm.

We then note that $V \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$, that supp $V \subset \bigcup_j$ supp $V_{n_j} \subset B_r(0)$, and $\|D^\alpha V\|_{L^2} < C_\alpha \,\forall \alpha$. So, $V \in \mathcal{V}_{r,d}$ thus $\{V_n\}$ has a convergent sub-sequence and $\mathcal{V}_{r,d}$ is compact.

□

The previous proposition gives that $\mathcal{V}_{r,d}$ is compact and we know that IsoRes $(V_0) \subset \mathcal{V}_{r,d}$, however we still need to show that IsoRes (V_0) is closed. To do this we will relate a function,$m_{V_i}(\lambda)$, that counts the order of the poles of the resolvent to $m_{D_{V_i}}(\lambda)$ which counts the zeroes of the p determinant of the resolvent. We can then use Hurwitz's theorem to show that the zeroes and thus poles are the same.

Definition 5.2.3. *We define the regularized p-determinant of an operator $A \in \mathcal{L}_p$ using the following operator*

$$R_p(A) = (I + A) \exp\left(- \sum_{i=1}^{p-1} \frac{(-A)^i}{i!} \right) - I.$$

The p-determinant is then defined to be

$$\det_p(I + A) := \det(I + R_p(A)).$$

Proposition 5.2.2. *If $d \leq 3, \mathrm{IsoRes}(V_0, r, d)$ is closed under the Fréchet metric.*

Proof. Let $\{V_i\} \subset \mathrm{IsoRes}(V_0, r, 3)$ be a convergent sequence in C^∞ such that $V_i \to V \in V_{r,3}$. Define

$$D_{V_i}(\lambda) = \det_p(I + V_i R_0(\lambda)\rho)$$

where $p = \frac{d+1}{2} = 2$, $\rho \in C_0^\infty$ and $\rho|_{B_r} = \chi_{B_r}$.

Let $m_{V_i}(\lambda)$ be the order of the pole of R_{V_i} at λ and $m_{D_{V_i}}(\lambda)$ be the order of zeroes for D_{V_i} then Zworski and Dyatlov [8] give

$$m_{V_i}(\lambda) = m_{D_{V_i}}(\lambda).$$

So it remains to show that $m_{D_{V_i}}(\lambda) \to m_{D_{V_\infty}}(\lambda)$ or $D_{V_i} \to D_{V_\infty}$ due to Hurwitz's Theorem 6.0.3. An important detail to note in regards to Hurwitz's theorem is the requirement that $D_V(\lambda) \not\equiv 0$. However we know the limiting function $V_\infty \in C_0^\infty(B_r(0))$ and thus by Zworski-Sá Baretto [10] $R_{V_\infty}(\lambda)$ has at most countable number of poles which implies $D_V(\lambda)$ has at most a countable number of zeroes. Thus, $D_V(\lambda) \not\equiv 0$ and we may apply Hurowitz's theorem once we show $D_{V_i} \to D_{V_\infty}$. As a remark we say at most coutable because if $V = 0$ then $D_0(\lambda) = \det_p(I) = \det(I) = 1$.

Let

$$K_i = V_i R_0 \rho$$

and

$$R_2(V_i) = (I + K_i)e^{-K_i} - I$$

then

$$\begin{aligned}
|D_{V_i} - D_{V_\infty}| &= |det(I + R_2(V_i)) - det(I + R_2(V_\infty))| \\
&\leq \|R_2(V_i) - R_2(V_\infty)\|_1 e^{1+\|R_2(V_i)\|_1 + \|R_2(V_\infty)\|_1} \qquad (5.8) \\
&\leq C_{V_0,\lambda} \|(I + K_i)e^{-K_i} - (I + K_\infty)e^{-K_\infty}\|_1.
\end{aligned}$$

Expanding we have

$$(I + K_i)e^{-K_i} = \sum_{m=0}^{\infty} \frac{(-K_i)^m}{m!} + \sum_{m=0}^{\infty} \frac{(-1)^m K_i^{m+1}}{m!} = \sum_{m=0}^{\infty} \frac{(1-m)(-K_i)^m}{m!}$$

$$(I + K_i)e^{-K_i} - (I + K_\infty)e^{-K_\infty} = \sum_{m=2}^{\infty} \frac{(1-m)}{m!}[(-K_i)^m - (-K_\infty)^m].$$

We then consider the terms $(-K_i)^m - (-K_\infty)^m$ and factor the difference as

$$K_i^m - K_\infty^m = \sum_{l=1}^{m} K_i^{l-1}(K_\infty - K_i)K_\infty^{m-l}.$$

Case 1: $n \geq 3, l \geq 2$:

$$\begin{aligned}
\|K_i^{l-1}(K_\infty - K_i)K_\infty^{n-l}\|_1 &\leq \|K_i^{l-1}\|_1 \|(K_\infty - K_i)K_\infty^{m-l}\| \\
&\leq C\|K_i\|^{l-1}(\sum_j j^{-\frac{2l}{3}})\|\delta V\|\|V_\infty\|^{m-l}\|R_0(\lambda)\|^{m-l+1} \\
&\leq C\|V_i\|^{l-1}\|R_0(\lambda)\|^{l-1}\frac{3}{2l-3}\|\delta V\|\|V_\infty\|^{m-l}\|R_0(\lambda)\|^{m-l+1} \\
&\leq C\frac{3}{2l-3}\|\delta V\|\|V_i\|^{l-1}\|V_\infty\|^{m-l}\|R_0(\lambda)\|^{m} \\
&\leq C_R C_{V_0}^{m-1}\|\delta V\|_\infty \|R_0(\lambda)\|^m \\
&\leq C_R C_{V_0}^{m-1}\|\delta V\|_\infty \langle\lambda\rangle^m e^{C(Im\lambda)m} \\
&\leq C_R\|\delta V\|_\infty \langle C_{V_0}\lambda\rangle^m e^{C(Im\lambda)m}.
\end{aligned} \tag{5.9}$$

Case 2: Assuming $d > 3, l < 2$ gives a similar estimate due to symmetry.

Case 3: Assume $d = 2, l = 0, 2$. This case follows from the previous cases.

Case 4: Assume $d = 2, l = 1$

Define $J = \rho R_0(\lambda)\rho$ which gives

$$K_i(K_\infty - K_i) = K_i \delta V J.$$

Then

$$\mu_{2j}(K_i\delta V J) \leq \mu_j(K_i)\mu_0(\delta V)\mu_j(J)$$

and

$$\mu_{2j+1}(K_i\delta V J) \leq \mu_j(K_i)\mu_0(\delta V)\mu_{j+1}(J) \leq \mu_j(K_i)\mu_0(\delta V)\mu_j(J).$$

Using the previous trace/singular value estimates gives

$$\|K_i(K_\infty - K_i)\|_1 \leq C_{R_0(\lambda),V_0}\|\delta V\|_\infty.$$

Thus,

$$\begin{aligned}\|K_i^m - K_\infty^m\|_1 &\le \sum_{l=1}^{m} \|K_i^{l-1}(K_\infty - K_i)K_\infty^{m-l}\|_1 \\ &\le m\|\delta V\|_\infty \langle C\lambda\rangle^m e^{C(Im\lambda)m}.\end{aligned} \tag{5.10}$$

Summing gives

$$\begin{aligned}\|(I+K_i)e^{-K_i} - (I+K_\infty)e^{-K_\infty}\|_1 &\le \sum_{m=2}^{\infty} \frac{(1-m)}{m!}\|(K_i)^m - (K_\infty)^m\|_1 \\ &\le \|\delta V\|_\infty \sum_{m=2}^{\infty} \frac{(1-m)me^{C(Im\lambda)m}}{m!}\end{aligned} \tag{5.11}$$

The ratio test show that the sum converges for all λ, and taking $\|\delta V\|_\infty \to 0$ gives convergence of the trace norm and so $D_{V_i} \to D_\infty$. □

We now have everything we need to proof the following compactness result.

Theorem 5.2.1. *Let $V \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$ with $d = 1, 3$, the operator*

$$H_V = -\Delta + V,$$

and fix $r > 0$ and $V_0 \in C_0^\infty(B_r(0), \mathbb{R})$. Then the set

$$\mathrm{IsoRes}(V_0) = \{V \in C_0^\infty(B_r(0), \mathbb{R}) \,|\, V \in \mathrm{IsoRes}(V_0)\}$$

is compact in the C^∞ topology.

Proof. Theorem 5.1.2 gives that $\mathrm{IsoRes}(V_0) \subset \mathcal{V}_{r,d}$, 5.2.1 gives that $\mathcal{V}_{r,d}$ is compact, and 5.2.2 gives that $\mathrm{IsoRes}(V_0)$ is closed under the Fréchet metric. Closed subsets of compact sets are compact, thus $\mathrm{IsoRes}(V_0)$ is compact.

□

Chapter 6 Appendix A

Sobolev Inequalites

Theorem 6.0.2. *General Sobolev Inequality*

Let U be a bounded open subset of $\mathbb{R}^d$ with C^1 boundary. Assume $u \in W_{k,p}$. Then if $k < \frac{d}{p}$ then $u \in L^q(U)$ where $\frac{1}{q} = \frac{1}{p} - \frac{k}{d}$.

Theorem 6.0.3. *If $u \in W^{k,2}(\mathbb{R}^d)$ for $k > \frac{d}{2}$ then $u \in L^\infty(\mathbb{R}^d)$ with*

$$\|u\|_\infty \leq C\|u\|_{W^{k,2}}$$

Proof. WLOG we identify u with its continuous representative in $W^{k,2}$. Then we rewrite u using the Fourier transform and apply Cauchy-Schwarz.

$$\begin{aligned} |u(x)| &= |\frac{1}{(2\pi)^{\frac{d}{2}}} \int e^{ixy}\hat{u}(y)\,dy| \\ &\leq |\frac{1}{(2\pi)^{\frac{d}{2}}} \int \frac{e^{ixy}}{1+|y|^k}(1+|y|^k)\hat{u}(y)\,dy| \\ &\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\int \frac{1}{(1+|y|^k)^2}\,dy\right)^{\frac{1}{2}} \|(1+|y|^k)\hat{u}\|_2 \end{aligned} \tag{6.1}$$

The Fourier characterization of $W^{k,2}(\mathbb{R}^d)$ then gives that

$$\|(1+|y|^k)\hat{u}\|_2 \leq C\|u\|_{W^{k,2}} \tag{6.2}$$

so we only need to show bounds on the first integral. Changing to polar coordinates we see that

$$\begin{aligned} \int \frac{1}{(1+|y|^k)^2}\,dy &= \int \int_0^\infty \frac{r^{d-1}}{(1+r^k)^2} dr d\sigma \\ &= \sigma(S^{d-1}) \left(1 + \int_1^\infty \frac{r^{d-1}}{(1+r^k)^2}\,dr\right) \\ &\leq \sigma(S^{d-1}) \left(1 + \int_1^\infty \frac{r^{d-1}}{r^{2k}}\,dr\right) \\ &= \sigma(S^{d-1}) \left(1 + \int_1^\infty r^{d-2k-1}\,dr\right). \end{aligned} \tag{6.3}$$

Since $k > \frac{d}{2}$ we can see the integral converges and the bound is proven.

□

Theorem 6.0.4. *Let $\Omega \subset \mathbb{R}^d$, $u \in W^{j+m,2}(\Omega)$ and $m = \frac{d}{2}$ then there exists a constant C such that*

$$\|\|u\|_{W^{j,q}(\Omega)} \leq C\|u\|_{W^{j+m,2}(\Omega)} \tag{6.4}$$

for $2 \leq q \leq \infty$.

This special case of the Sobolev Imbedding theorem comes from Adams and Fournier [5].

Lemma 6.0.2. *Mean Value Theorem*

Let $\Omega \subset \mathbb{R}^d$ be open and simply connected. Then for $u \in C^1(\Omega)$, there exists an α such that $w = \alpha x + (1-\alpha)y$

$$|u(x) - u(y)| \leq |\nabla u(w) \cdot (x-y)|$$

Lemma 6.0.3. *Hurwitz's Theorem [11]*

Let $G \subset \mathbb{C}$ be open and $\{f_n\}$ be a sequence of analytic functions on G such that $f_n \to f$. If $f \not\equiv 0, \bar{B}_R(a) \subset G$, and $f(z) \neq 0$ for $|z-a| = R$ then there is an N such that that $n \geq N$ implies f and f_n have the same number of zeros in $B_R(a)$.

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